# Exercises for Elementary Differential Geometry

## Chapter 1

1.1.1 Is $\boldsymbol{\gamma}(t) = (t^2, t^4)$ a parametrization of the parabola $y = x^2$?

1.1.2 Find parametrizations of the following level curves:
(i) $y^2 - x^2 = 1$.
(ii) $\frac{x^2}{4} + \frac{y^2}{9} = 1$.

1.1.3 Find the Cartesian equations of the following parametrized curves:
(i) $\boldsymbol{\gamma}(t) = (\cos^2 t, \sin^2 t)$.
(ii) $\boldsymbol{\gamma}(t) = (e^t, t^2)$.

1.1.4 Calculate the tangent vectors of the curves in Exercise 1.1.3.

1.1.5 Sketch the astroid in Example 1.1.4. Calculate its tangent vector at each point. At which points is the tangent vector zero?

1.1.6 Consider the ellipse

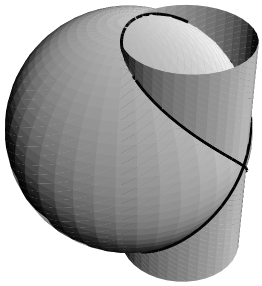
$$\frac{x^2}{p^2} + \frac{y^2}{q^2} = 1,$$

where $p > q > 0$. The *eccentricity* of the ellipse is $\epsilon = \sqrt{1 - \frac{q^2}{p^2}}$ and the points $(\pm\epsilon p, 0)$ on the $x$-axis are called the *foci* of the ellipse, which we denote by $\mathbf{f}_1$ and $\mathbf{f}_2$. Verify that $\boldsymbol{\gamma}(t) = (p\cos t, q\sin t)$ is a parametrization of the ellipse. Prove that:
(i) The sum of the distances from $\mathbf{f}_1$ and $\mathbf{f}_2$ to any point $\mathbf{p}$ on the ellipse does not depend on $\mathbf{p}$.
(ii) The product of the distances from $\mathbf{f}_1$ and $\mathbf{f}_2$ to the tangent line at any point $\mathbf{p}$ of the ellipse does not depend on $\mathbf{p}$.
(iii) If $\mathbf{p}$ is any point on the ellipse, the line joining $\mathbf{f}_1$ and $\mathbf{p}$ and that joining $\mathbf{f}_2$ and $\mathbf{p}$ make equal angles with the tangent line to the ellipse at $\mathbf{p}$.

1.1.7 A *cycloid* is the plane curve traced out by a point on the circumference of a circle as it rolls without slipping along a straight line. Show that, if the straight line is the $x$-axis and the circle has radius $a > 0$, the cycloid can be parametrized as

$$\boldsymbol{\gamma}(t) = a(t - \sin t, 1 - \cos t).$$

1.1.8 Show that $\boldsymbol{\gamma}(t) = (\cos^2 t - \frac{1}{2}, \sin t\cos t, \sin t)$ is a parametrization of the curve of intersection of the circular cylinder of radius $\frac{1}{2}$ and axis the $z$-axis with the sphere of radius 1 and centre $(-\frac{1}{2}, 0, 0)$. This is called *Viviani's Curve* - see below.



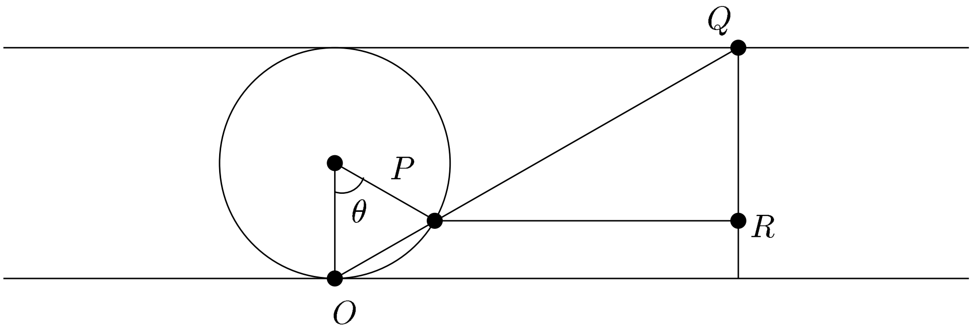
1.1.9 The *normal line* to a curve at a point $\mathbf{p}$ is the straight line passing through $\mathbf{p}$ perpendicular to the tangent line at $\mathbf{p}$. Find the tangent and normal lines to the curve $\boldsymbol{\gamma}(t) = (2\cos t - \cos 2t, 2\sin t - \sin 2t)$ at the point corresponding to $t = \pi/4$.

1.1.10 Find parametrizations of the following level curves:
(i) $y^2 = x^2(x^2 - 1)$.
(ii) $x^3 + y^3 = 3xy$ (the *folium of Descartes*).

1.1.11 Find the Cartesian equations of the following parametrized curves:
(i) $\boldsymbol{\gamma}(t) = (1 + \cos t, \sin t(1 + \cos t))$.
(ii) $\boldsymbol{\gamma}(t) = (t^2 + t^3, t^3 + t^4)$.

1.1.12 Calculate the tangent vectors of the curves in Exercise 1.1.11. For each curve, determine at which point(s) the tangent vector vanishes.

1.1.13 If $P$ is any point on the circle $\mathcal{C}$ in the $xy$-plane of radius $a > 0$ and centre $(0, a)$, let the straight line through the origin and $P$ intersect the line $y = 2a$ at $Q$, and let the line through $P$ parallel to the $x$-axis intersect the line through $Q$ parallel to the $y$-axis at $R$. As $P$ moves around $\mathcal{C}$, $R$ traces out a curve called the *witch of Agnesi*. For this curve, find
(i) a parametrization;
(ii) its Cartesian equation.




1.1.14 Generalize Exercise 1.1.7 by finding parametrizations of an *epicycloid* (resp. *hypocycloid*), the curve traced out by a point on the circumference of a circle as it rolls without slipping around the outside (resp. inside) of a fixed circle.

1.1.15 For the *logarithmic spiral* $\boldsymbol{\gamma}(t) = (e^t \cos t, e^t \sin t)$, show that the angle between $\boldsymbol{\gamma}(t)$ and the tangent vector at $\boldsymbol{\gamma}(t)$ is independent of $t$. (There is a picture of the logarithmic spiral in Example 1.2.2.)

1.1.16 Show that all the normal lines to the curve

$$\boldsymbol{\gamma}(t) = (\cos t + t \sin t, \sin t - t \cos t)$$

are the same distance from the origin.

1.2.1 Calculate the arc-length of the *catenary* $\boldsymbol{\gamma}(t) = (t, \cosh t)$ starting at the point $(0, 1)$. This curve has the shape of a heavy chain suspended at its ends - see Exercise 2.2.4.

1.2.2 Show that the following curves are unit-speed:
(i) $\boldsymbol{\gamma}(t) = \left(\frac{1}{3}(1+t)^{3/2}, \frac{1}{3}(1-t)^{3/2}, \frac{t}{\sqrt{2}}\right)$.
(ii) $\boldsymbol{\gamma}(t) = \left(\frac{4}{5}\cos t, 1 - \sin t, -\frac{3}{5}\cos t\right)$.

1.2.3 A plane curve is given by

$$\boldsymbol{\gamma}(\theta) = (r \cos \theta, r \sin \theta),$$

where $r$ is a smooth function of $\theta$ (so that $(r, \theta)$ are the polar coordinates of $\boldsymbol{\gamma}(\theta)$). Under what conditions is $\boldsymbol{\gamma}$ regular? Find all functions $r(\theta)$ for which $\boldsymbol{\gamma}$ is unit-speed. Show that, if $\boldsymbol{\gamma}$ is unit-speed, the image of $\boldsymbol{\gamma}$ is a circle; what is its radius?

1.2.4 This exercise shows that *a straight line is the shortest curve joining two given points*. Let $\mathbf{p}$ and $\mathbf{q}$ be the two points, and let $\boldsymbol{\gamma}$ be a curve passing through both, say $\boldsymbol{\gamma}(a) = \mathbf{p}$, $\boldsymbol{\gamma}(b) = \mathbf{q}$, where $a < b$. Show that, if $\mathbf{u}$ is any unit vector,

$$\dot{\boldsymbol{\gamma}}.\mathbf{u} \leq \parallel \dot{\boldsymbol{\gamma}} \parallel$$

and deduce that

$$(\mathbf{q} - \mathbf{p}).\mathbf{u} \leq \int_a^b \parallel \dot{\boldsymbol{\gamma}} \parallel \, dt.$$

By taking $\mathbf{u} = (\mathbf{q} - \mathbf{p}) / \parallel \mathbf{q} - \mathbf{p} \parallel$, show that the length of the part of $\boldsymbol{\gamma}$ between $\mathbf{p}$ and $\mathbf{q}$ is at least the straight line distance $\parallel \mathbf{q} - \mathbf{p} \parallel$.

1.2.5 Find the arc-length of the curve

$$\boldsymbol{\gamma}(t) = (3t^2, t - 3t^3)$$

starting at $t = 0$.

1.2.6 Find, for $0 \leq x \leq \pi$, the arc-length of the segment of the curve

$$\boldsymbol{\gamma}(t) = (2\cos t - \cos 2t, 2\sin t - \sin 2t)$$

corresponding to $0 \leq t \leq x$.

1.2.7 Calculate the arc-length along the cycloid in Exercise 1.1.7 corresponding to one complete revolution of the circle.

1.2.8 Calculate the length of the part of the curve

$$\boldsymbol{\gamma}(t) = (\sinh t - t, 3 - \cosh t)$$

cut off by the $x$-axis.

1.2.9 Show that a curve $\boldsymbol{\gamma}$ such that $\dddot{\boldsymbol{\gamma}} = \mathbf{0}$ everywhere is contained in a plane.

1.3.1 Which of the following curves are regular?
(i) $\boldsymbol{\gamma}(t) = (\cos^2 t, \sin^2 t)$ for $t \in \mathbb{R}$.
(ii) the same curve as in (i), but with $0 < t < \pi/2$.
(iii) $\boldsymbol{\gamma}(t) = (t, \cosh t)$ for $t \in \mathbb{R}$.
Find unit-speed reparametrizations of the regular curve(s).

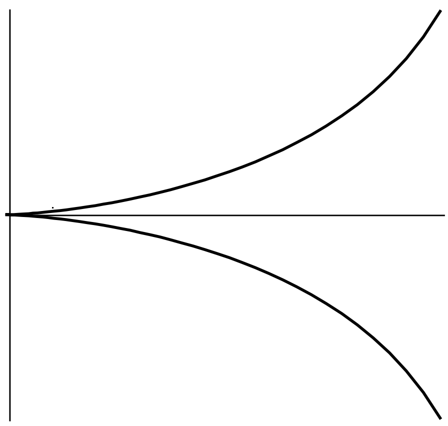
1.3.2 The *cissoid of Diocles* (see below) is the curve whose equation in terms of polar coordinates $(r, \theta)$ is

$$r = \sin\theta \tan\theta, \quad -\pi/2 < \theta < \pi/2.$$

Write down a parametrization of the cissoid using $\theta$ as a parameter and show that

$$\boldsymbol{\gamma}(t) = \left(t^2, \frac{t^3}{\sqrt{1-t^2}}\right), \quad -1 < t < 1,$$

is a reparametrization of it.



1.3.3 The simplest type of singular point of a curve $\boldsymbol{\gamma}$ is an *ordinary cusp*: a point $\mathbf{p}$ of $\boldsymbol{\gamma}$, corresponding to a parameter value $t_0$, say, is an ordinary cusp if $\dot{\boldsymbol{\gamma}}(t_0) = \mathbf{0}$ and the vectors $\ddot{\boldsymbol{\gamma}}(t_0)$ and $\dddot{\boldsymbol{\gamma}}(t_0)$ are linearly independent (in particular, these vectors must both be non-zero). Show that:
(i) the curve $\boldsymbol{\gamma}(t) = (t^m, t^n)$, where $m$ and $n$ are positive integers, has an ordinary cusp at the origin if and only if $(m, n) = (2, 3)$ or $(3, 2)$;
(ii) the cissoid in Exercise 1.3.2 has an ordinary cusp at the origin;
(iii) if $\boldsymbol{\gamma}$ has an ordinary cusp at a point $\mathbf{p}$, so does any reparametrization of $\boldsymbol{\gamma}$.

1.3.4 Show that:
(i) if $\tilde{\boldsymbol{\gamma}}$ is a reparametrization of a curve $\boldsymbol{\gamma}$, then $\boldsymbol{\gamma}$ is a reparametrization of $\tilde{\boldsymbol{\gamma}}$;
(ii) if $\tilde{\boldsymbol{\gamma}}$ is a reparametrization of $\boldsymbol{\gamma}$, and $\hat{\boldsymbol{\gamma}}$ is a reparametrization of $\tilde{\boldsymbol{\gamma}}$, then $\hat{\boldsymbol{\gamma}}$ is a reparametrization of $\boldsymbol{\gamma}$.

1.3.5 Repeat Exercise 1.3.1 for the following curves:
(i) $\boldsymbol{\gamma}(t) = (t^2, t^3)$, $t \in \mathbb{R}$.
(ii) $\boldsymbol{\gamma}(t) = ((1 + \cos t)\cos t, (1 + \cos t)\sin t)$, $-\pi < t < \pi$.

1.3.6 Show that the curve

$$\boldsymbol{\gamma}(t) = \left(2t, \frac{2}{1+t^2}\right), \qquad t > 0,$$

is regular and that it is a reparametrization of the curve

$$\tilde{\boldsymbol{\gamma}}(t) = \left(\frac{2\cos t}{1+\sin t}, 1 + \sin t\right), \qquad -\frac{\pi}{2} < t < \frac{\pi}{2}.$$

1.3.7 The curve

$$\boldsymbol{\gamma}(t) = (a \sin \omega t, b \sin t),$$

where $a, b$ and $\omega$ are non-zero constants, is called a *Lissajous figure*. Show that $\boldsymbol{\gamma}$ is regular if and only if $\omega$ is not the quotient of two odd integers.

1.3.8 Let $\boldsymbol{\gamma}$ be a curve in $\mathbb{R}^n$ and let $\tilde{\boldsymbol{\gamma}}$ be a reparametrization of $\boldsymbol{\gamma}$ with reparametrization map $\phi$ (so that $\tilde{\boldsymbol{\gamma}}(\tilde{t}) = \boldsymbol{\gamma}(\phi(\tilde{t}))$). Let $\tilde{t}_0$ be a fixed value of $\tilde{t}$ and let $t_0 = \phi(\tilde{t}_0)$. Let $s$ and $\tilde{s}$ be the arc-lengths of $\boldsymbol{\gamma}$ and $\tilde{\boldsymbol{\gamma}}$ starting at the point $\boldsymbol{\gamma}(t_0) = \tilde{\boldsymbol{\gamma}}(\tilde{t}_0)$. Prove that $\tilde{s} = s$ if $d\phi/d\tilde{t} > 0$ for all $\tilde{t}$, and $\tilde{s} = -s$ if $d\phi/d\tilde{t} < 0$ for all $\tilde{t}$.

1.3.9 Suppose that all the tangent lines of a regular plane curve pass through some fixed point. Prove that the curve is part of a straight line. Prove the same result if all the normal lines are parallel.

1.4.1 Show that the *Cayley sextic*

$$\boldsymbol{\gamma}(t) = (\cos^3 t \cos 3t, \cos^3 t \sin 3t), \qquad t \in \mathbb{R},$$

is a closed curve which has exactly one self-intersection. What is its period? (The name of this curve derives from the fact that its Cartesian equation involves a polynomial of degree six.)

1.4.2 Give an example to show that a reparametrization of a closed curve need not be closed.

1.4.3 Show that if a curve $\boldsymbol{\gamma}$ is $T_1$-periodic and $T_2$-periodic, it is $(k_1T_1+k_2T_2)$-periodic for any integers $k_1, k_2$.

1.4.4 Let $\boldsymbol{\gamma} : \mathbb{R} \to \mathbb{R}^n$ be a curve and suppose that $T_0$ is the smallest positive number such that $\boldsymbol{\gamma}$ is $T_0$-periodic. Prove that $\boldsymbol{\gamma}$ is $T$-periodic if and only if $T = kT_0$ for some integer $k$.

1.4.5 Suppose that a *non-constant* function $\gamma : \mathbb{R} \to \mathbb{R}$ is $T$-periodic for some $T \neq 0$. This exercise shows that there is a *smallest* positive $T_0$ such that $\gamma$ is $T_0$-periodic. The proof uses a little real analysis. Suppose for a contradiction that there is no such $T_0$.
(i) Show that there is a sequence $T_1, T_2, T_3, \ldots$ such that $T_1 > T_2 > T_3 > \cdots > 0$ and that $\gamma$ is $T_r$-periodic for all $r \geq 1$.
(ii) Show that the sequence $\{T_r\}$ in (i) can be chosen so that $T_r \to 0$ as $r \to \infty$.
(iii) Show that the existence of a sequence $\{T_r\}$ as in (i) such that $T_r \to 0$ as $r \to \infty$ implies that $\gamma$ is constant.

1.4.6 Let $\boldsymbol{\gamma} : \mathbb{R} \to \mathbb{R}^n$ be a non-constant curve that is $T$-periodic for some $T > 0$. Show that $\boldsymbol{\gamma}$ is closed.

1.4.7 Show that the following curve is not closed and that it has exactly one self-intersection:

$$\boldsymbol{\gamma}(t) = \left(\frac{t^2-3}{t^2+1}, \frac{t(t^2-3)}{t^2+1}\right).$$

1.4.8 Show that the curve

$$\boldsymbol{\gamma}(t) = ((2+\cos t)\cos\omega t, (2+\cos t)\sin\omega t, \sin t),$$

where $\omega$ is a constant, is closed if and only if $\omega$ is a rational number. Show that, if $\omega = m/n$ where $m$ and $n$ are integers with no common factor, the period of $\boldsymbol{\gamma}$ is $2\pi n$.

1.5.1 Show that the curve $\mathcal{C}$ with Cartesian equation

$$y^2 = x(1-x^2)$$

is not connected. For what range of values of $t$ is

$$\boldsymbol{\gamma}(t) = (t, \sqrt{t-t^3})$$

a parametrization of $\mathcal{C}$? What is the image of this parametrization?

1.5.2 State an analogue of Theorem 1.5.1 for level curves in $\mathbb{R}^3$ given by $f(x,y,z) = g(x,y,z) = 0$.

1.5.3 State and prove an analogue of Theorem 1.5.2 for curves in $\mathbb{R}^3$ (or even $\mathbb{R}^n$). (This is easy.)

1.5.4 Show that the *conchoid*

$$(x-1)^2(x^2+y^2) = x^2$$

is not connected, but is the union of two disjoint connected curves (consider the line $x = 1$). How do you reconcile this with its (single) parametrization

$$\boldsymbol{\gamma}(t) = (1+\cos t, \sin t + \tan t)\,?$$

1.5.5 Show that the condition on $f$ and $g$ in Exercise 1.5.2 is satisfied for the level curve given by

$$x^2+y^2 = \frac{1}{4}, \quad x^2+y^2+z^2+x = \frac{3}{4}$$

*except* at the point $(1/2, 0, 0)$. Note that Exercise 1.1.15 gives a parametrization $\boldsymbol{\gamma}$ of this level curve; is $(1/2, 0, 0)$ a singular point of $\boldsymbol{\gamma}$?

1.5.6 Sketch the level curve $\mathcal{C}$ given by $f(x,y) = 0$ when $f(x,y) = y - |x|$. Note that $f$ does *not* satisfy the conditions in Theorem 1.5.1 because $\partial f/\partial x$ does not exist at the point $(0,0)$ on the curve. Show nevertheless that there is a smooth parametrized curve $\boldsymbol{\gamma}$ whose image is the whole of $\mathcal{C}$. Is there a *regular* parametrized curve with this property?

# Chapter 2

2.1.1 Compute the curvature of the following curves:
(i) $\boldsymbol{\gamma}(t) = \left(\frac{1}{3}(1+t)^{3/2}, \frac{1}{3}(1-t)^{3/2}, \frac{t}{\sqrt{2}}\right)$.
(ii) $\boldsymbol{\gamma}(t) = \left(\frac{4}{5}\cos t, 1-\sin t, -\frac{3}{5}\cos t\right)$.
(iii) $\boldsymbol{\gamma}(t) = (t, \cosh t)$.
(iv) $\boldsymbol{\gamma}(t) = (\cos^3 t, \sin^3 t)$.
For the astroid in (iv), show that the curvature tends to $\infty$ as we approach one of the points $(\pm 1, 0)$, $(0, \pm 1)$. Compare with the sketch found in Exercise 1.1.5.

2.1.2 Show that, if the curvature $\kappa(t)$ of a regular curve $\boldsymbol{\gamma}(t)$ is $> 0$ everywhere, then $\kappa(t)$ is a smooth function of $t$. Give an example to show that this may not be the case without the assumption that $\kappa > 0$.

2.1.3 Show that the curvature of the curve

$$\boldsymbol{\gamma}(t) = (t - \sinh t \cosh t, 2\cosh t), \quad t > 0,$$

is never zero, but that it tends to zero as $t \to \infty$.

2.1.4 Show that the curvature of the curve

$$\boldsymbol{\gamma}(t) = (\sec t, \sec t \tan t), \qquad -\pi/2 < t < \pi/2,$$

vanishes at exactly two points on the curve.

2.2.1 Show that, if $\boldsymbol{\gamma}$ is a unit-speed plane curve,

$$\dot{\mathbf{n}}_s = -\kappa_s \mathbf{t}.$$

2.2.2 Show that the signed curvature of any regular plane curve $\boldsymbol{\gamma}(t)$ is a smooth function of $t$. (Compare with Exercise 2.1.2.)

2.2.3 Let $\boldsymbol{\gamma}$ and $\tilde{\boldsymbol{\gamma}}$ be two plane curves. Show that, if $\tilde{\boldsymbol{\gamma}}$ is obtained from $\boldsymbol{\gamma}$ by applying an isometry $M$ of $\mathbb{R}^2$, the signed curvatures $\kappa_s$ and $\tilde{\kappa}_s$ of $\boldsymbol{\gamma}$ and $\tilde{\boldsymbol{\gamma}}$ are equal if $M$ is direct but that $\tilde{\kappa}_s = -\kappa_s$ if $M$ is opposite (in particular, $\boldsymbol{\gamma}$ and $\tilde{\boldsymbol{\gamma}}$ have the same curvature). Show, conversely, that if $\boldsymbol{\gamma}$ and $\tilde{\boldsymbol{\gamma}}$ have the same nowhere-vanishing curvature, then $\tilde{\boldsymbol{\gamma}}$ can be obtained from $\boldsymbol{\gamma}$ by applying an isometry of $\mathbb{R}^2$.

2.2.4 Let $k$ be the signed curvature of a plane curve $\mathcal{C}$ expressed in terms of its arc-length. Show that, if $\mathcal{C}_a$ is the image of $\mathcal{C}$ under the dilation $\mathbf{v} \mapsto a\mathbf{v}$ of the plane (where $a$ is a non-zero constant), the signed curvature of $\mathcal{C}_a$ in terms of *its* arc-length $s$ is $\frac{1}{a}k(\frac{s}{a})$.
A heavy chain suspended at its ends hanging loosely takes the form of a plane curve $\mathcal{C}$. Show that, if $s$ is the arc-length of $\mathcal{C}$ measured from its lowest point, $\varphi$ the angle between the tangent of $\mathcal{C}$ and the horizontal, and $T$ the tension in the chain, then

$$T \cos\varphi = \lambda, \quad T \sin\varphi = \mu s,$$

where $\lambda, \mu$ are non-zero constants (we assume that the chain has constant mass per unit length). Show that the signed curvature of $\mathcal{C}$ is

$$\kappa_s = \frac{1}{a}\left(1 + \frac{s^2}{a^2}\right)^{-1},$$

where $a = \lambda/\mu$, and deduce that $\mathcal{C}$ can be obtained from the catenary in Example 2.2.4 by applying a dilation and an isometry of the plane.

2.2.5 Let $\boldsymbol{\gamma}(t)$ be a regular plane curve and let $\lambda$ be a constant. The *parallel curve* $\boldsymbol{\gamma}^\lambda$ of $\boldsymbol{\gamma}$ is defined by

$$\boldsymbol{\gamma}^\lambda(t) = \boldsymbol{\gamma}(t) + \lambda \mathbf{n}_s(t).$$

Show that, if $\lambda\kappa_s(t) \neq 1$ for all values of $t$, then $\boldsymbol{\gamma}^\lambda$ is a regular curve and that its signed curvature is $\kappa_s/|1 - \lambda\kappa_s|$.

2.2.6 Another approach to the curvature of a unit-speed plane curve $\boldsymbol{\gamma}$ at a point $\boldsymbol{\gamma}(s_0)$ is to look for the 'best approximating circle' at this point. We can then *define* the curvature of $\boldsymbol{\gamma}$ to be the reciprocal of the radius of this circle.
Carry out this programme by showing that the centre of the circle which passes through three nearby points $\boldsymbol{\gamma}(s_0)$ and $\boldsymbol{\gamma}(s_0 \pm \delta s)$ on $\boldsymbol{\gamma}$ approaches the point

$$\boldsymbol{\epsilon}(s_0) = \boldsymbol{\gamma}(s_0) + \frac{1}{\kappa_s(s_0)}\mathbf{n}_s(s_0)$$

as $\delta s$ tends to zero. The circle $\mathcal{C}$ with centre $\boldsymbol{\epsilon}(s_0)$ passing through $\boldsymbol{\gamma}(s_0)$ is called the *osculating circle* to $\boldsymbol{\gamma}$ at the point $\boldsymbol{\gamma}(s_0)$, and $\boldsymbol{\epsilon}(s_0)$ is called the *centre of curvature* of $\boldsymbol{\gamma}$ at $\boldsymbol{\gamma}(s_0)$. The radius of $\mathcal{C}$ is $1/|\kappa_s(s_0)| = 1/\kappa(s_0)$, where $\kappa$ is the curvature of $\boldsymbol{\gamma}$ - this is called the *radius of curvature* of $\mathcal{C}$ at $\boldsymbol{\gamma}(s_0)$.

2.2.7 With the notation in the preceding exercise, we regard $\boldsymbol{\epsilon}$ as the parametrization of a new curve, called the *evolute* of $\boldsymbol{\gamma}$ (if $\boldsymbol{\gamma}$ is any regular plane curve, its evolute is defined to be that of a unit-speed reparametrization of $\boldsymbol{\gamma}$). Assume that $\dot{\kappa}_s(s) \neq 0$ for all values of $s$ (a dot denoting $d/ds$), say $\dot{\kappa}_s > 0$ for all $s$ (this can be achieved by replacing $s$ by $-s$ if necessary). Show that the arc-length of $\boldsymbol{\epsilon}$ is $-\frac{1}{\kappa_s(s)}$ (up to adding a constant), and calculate the signed curvature of $\boldsymbol{\epsilon}$. Show also that all the normal lines to $\boldsymbol{\gamma}$ are tangent to $\boldsymbol{\epsilon}$ (for this reason, the evolute of $\boldsymbol{\gamma}$ is sometimes described as the 'envelope' of the normal lines to $\boldsymbol{\gamma}$).

Show that the evolute of the cycloid

$$\boldsymbol{\gamma}(t) = a(t - \sin t, 1 - \cos t), \qquad 0 < t < 2\pi,$$

where $a > 0$ is a constant, is

$$\boldsymbol{\epsilon}(t) = a(t + \sin t, -1 + \cos t)$$

(see Exercise 1.1.7) and that, after a suitable reparametrization, $\boldsymbol{\epsilon}$ can be obtained from $\boldsymbol{\gamma}$ by a translation of the plane.

2.2.8 A string of length $\ell$ is attached to the point $\boldsymbol{\gamma}(0)$ of a unit-speed plane curve $\boldsymbol{\gamma}(s)$. Show that when the string is wound onto the curve while being kept taught, its endpoint traces out the curve

$$\boldsymbol{\iota}(s) = \boldsymbol{\gamma}(s) + (\ell - s)\dot{\boldsymbol{\gamma}}(s),$$

where $0 < s < \ell$ and a dot denotes $d/ds$. The curve $\boldsymbol{\iota}$ is called the *involute* of $\boldsymbol{\gamma}$ (if $\boldsymbol{\gamma}$ is any regular plane curve, we define its involute to be that of a unit-speed reparametrization of $\boldsymbol{\gamma}$). Suppose that the signed curvature $\kappa_s$ of $\boldsymbol{\gamma}$ is never zero, say $\kappa_s(s) > 0$ for all $s$. Show that the signed curvature of $\boldsymbol{\iota}$ is $1/(\ell - s)$.

2.2.9 Show that the involute of the catenary

$$\boldsymbol{\gamma}(t) = (t, \cosh t)$$

with $l = 0$ (see the preceding exercise) is the *tractrix*

$$x = \cosh^{-1}\left(\frac{1}{y}\right) - \sqrt{1 - y^2}.$$

See §8.3 for a simple geometric characterization of this curve.

2.2.10 A unit-speed plane curve $\boldsymbol{\gamma}(s)$ rolls without slipping along a straight line $\ell$ parallel to a unit vector $\mathbf{a}$, and initially touches $\ell$ at a point $\mathbf{p} = \boldsymbol{\gamma}(0)$. Let $\mathbf{q}$ be a point fixed *relative to* $\boldsymbol{\gamma}$. Let $\boldsymbol{\Gamma}(s)$ be the point to which $\mathbf{q}$ has moved when $\boldsymbol{\gamma}$ has rolled a distance $s$ along $\ell$ (note that $\boldsymbol{\Gamma}$ will not usually be unit-speed). Let $\theta(s)$ be the angle between $\mathbf{a}$ and the tangent vector $\dot{\boldsymbol{\gamma}}$. Show that

$$\boldsymbol{\Gamma}(s) = \mathbf{p} + s\mathbf{a} + \rho_{-\theta(s)}(\mathbf{q} - \boldsymbol{\gamma}(s)),$$

where $\rho_\varphi$ is the rotation about the origin through an angle $\varphi$. Show further that

$$\dot{\boldsymbol{\Gamma}}(s).\rho_{-\theta(s)}(\mathbf{q} - \boldsymbol{\gamma}(s)) = 0.$$

Geometrically, this means that a point on $\boldsymbol{\Gamma}$ moves as if it is rotating about the instantaneous point of contact of the rolling curve with $\ell$. See Exercise 1.1.7 for a special case.

2.2.11 Show that, if two plane curves $\boldsymbol{\gamma}(t)$ and $\tilde{\boldsymbol{\gamma}}(t)$ have the same non-zero curvature for all values of $t$, then $\tilde{\boldsymbol{\gamma}}$ can be obtained from $\boldsymbol{\gamma}$ by applying an isometry of $\mathbb{R}^2$.

2.2.12 Show that if all the normal lines to a plane curve pass through some fixed point, the curve is part of a circle.

2.2.13 Let $\boldsymbol{\gamma}(t) = (e^{kt}\cos t, e^{kt}\sin t)$, where $-\infty < t < \infty$ and $k$ is a non-zero constant (a logarithmic spiral – see Example 1.2.2). Show that there is a unique unit-speed parameter $s$ on $\boldsymbol{\gamma}$ such that $s > 0$ for all $t$ and $s \to 0$ as $t \to \mp\infty$ if $\pm k > 0$, and express $s$ as a function of $t$.
Show that the signed curvature of $\boldsymbol{\gamma}$ is $1/ks$. Conversely, describe every curve whose signed curvature, as a function of arc-length $s$, is $1/ks$ for some non-zero constant $k$.

2.2.14 If $\boldsymbol{\gamma}$ is a plane curve, its *pedal curve* with respect to a fixed point $\mathbf{p}$ is the curve traced out by the foot of the perpendicular from $\mathbf{p}$ to the tangent line at a variable point of the curve. If $\boldsymbol{\gamma}$ is unit-speed, show that the pedal curve is parametrized by

$$\boldsymbol{\delta} = \mathbf{p} + ((\boldsymbol{\gamma} - \mathbf{p}).\mathbf{n}_s)\mathbf{n}_s,$$

where $\mathbf{n}_s$ is the signed unit normal of $\boldsymbol{\gamma}$. Show that $\boldsymbol{\delta}$ is regular if and only if $\boldsymbol{\gamma}$ has nowhere vanishing curvature and does not pass through $\mathbf{p}$.
Show that the pedal curve of the circle $\boldsymbol{\gamma}(t) = (\cos t, \sin t)$ with respect to the point $(-2, 0)$ is obtained by applying a translation to the limaçon in Example 1.1.7.

2.2.15 A unit-speed plane curve $\boldsymbol{\gamma}$ has the property that its tangent vector $\mathbf{t}(s)$ makes a fixed angle $\theta$ with $\boldsymbol{\gamma}(s)$ for all $s$. Show that:
(i) If $\theta = 0$, then $\boldsymbol{\gamma}$ is part of a straight line.
(ii) If $\theta = \pi/2$, then $\boldsymbol{\gamma}$ is a circle.
(iii) If $0 < \theta < \pi/2$, then $\boldsymbol{\gamma}$ is a logarithmic spiral.

2.2.16 Let $\boldsymbol{\gamma}^\lambda$ be a parallel curve of the parabola $\boldsymbol{\gamma}(t) = (t, t^2)$. Show that:
(i) $\boldsymbol{\gamma}^\lambda$ is regular if and only if $\lambda < 1/2$.
(ii) If $\lambda > 1/2$, $\boldsymbol{\gamma}^\lambda$ has exactly two singular points.
What happens if $\lambda = 1/2$?

2.2.17 This exercise gives another approach to the definition of the 'best approximating circle' to a curve $\boldsymbol{\gamma}$ at a point $\boldsymbol{\gamma}(t_0)$ of $\boldsymbol{\gamma}$ - see Exercise 2.2.6. We assume that $\boldsymbol{\gamma}$ is unit-speed for simplicity.
Let $\mathcal{C}$ be the circle with centre $\mathbf{c}$ and radius $R$, and consider the smooth function

$$F(t) = \parallel \boldsymbol{\gamma}(t) - \mathbf{c} \parallel^2 - R^2.$$

Show that $F(t_0) = \dot{F}(t_0) = 0$ if and only if $\mathcal{C}$ is tangent to $\boldsymbol{\gamma}$ at $\boldsymbol{\gamma}(t_0)$. This suggests that the 'best' approximating circle can be defined by the three conditions $F(t_0) = \dot{F}(t_0) = \ddot{F}(t_0) = 0$. Show that, if $\ddot{\boldsymbol{\gamma}}(t_0) \neq \mathbf{0}$, the unique circle $\mathcal{C}$ for which $F$ satisfies these conditions is the osculating circle to $\boldsymbol{\gamma}$ at the point $\boldsymbol{\gamma}(t_0)$.

2.2.18 Show that the evolute of the parabola $\boldsymbol{\gamma}(t) = (t, t^2)$ is the *semi-cubical parabola* $\boldsymbol{\epsilon}(t) = \left(-4t^3, 3t^2 + \frac{1}{2}\right)$.

2.2.19 Show that the evolute of the ellipse $\boldsymbol{\gamma}(t) = (a \cos t, b \sin t)$, where $a > b > 0$ are constants, is the astroid

$$\boldsymbol{\epsilon}(t) = \left( \frac{a^2 - b^2}{a} \cos^3 t, \frac{b^2 - a^2}{b} \sin^3 t \right).$$

(Compare Example 1.1.4.)

2.2.20 Show that all the parallel curves (Exercise 2.2.5) of a given curve have the same evolute.

2.2.21 Let $\boldsymbol{\gamma}$ be a regular plane curve. Show that:
(i) The involute of the evolute of $\boldsymbol{\gamma}$ is a parallel curve of $\boldsymbol{\gamma}$.
(ii) The evolute of the involute of $\boldsymbol{\gamma}$ is $\boldsymbol{\gamma}$.
(These statements might be compared to the fact that the integral of the derivative of a smooth function $f$ is equal to $f$ plus a constant, while the derivative of the integral of $f$ is $f$.)

2.2.22 A closed plane curve $\boldsymbol{\gamma}$ is parametrized by the direction of its normal lines, i.e. $\boldsymbol{\gamma}(\theta)$ is a $2\pi$-periodic curve such that $\theta$ is the angle between the normal line at $\boldsymbol{\gamma}(\theta)$ and the positive $x$-axis. Let $p(\theta)$ be the distance from the origin to the tangent line at $\boldsymbol{\gamma}(\theta)$. Show that:
(i) $\boldsymbol{\gamma}(\theta) = \left(p\cos\theta - \frac{dp}{d\theta}\sin\theta, p\sin\theta + \frac{dp}{d\theta}\cos\theta\right)$.
(ii) $\boldsymbol{\gamma}$ is regular if and only if $p + \frac{d^2p}{d\theta^2} > 0$ for all $\theta$ (we assume that this condition holds in the remainder of this exercise).
(iii) The signed curvature of $\boldsymbol{\gamma}$ is $\kappa_s = \left(p + \frac{d^2p}{d\theta^2}\right)^{-1}$.
(iv) The length of $\boldsymbol{\gamma}$ is $\int_0^{2\pi} p(\theta)d\theta$.
(v) The tangent lines at the points $\boldsymbol{\gamma}(\theta)$ and $\boldsymbol{\gamma}(\theta+\pi)$ are parallel and a distance $w(\theta) = p(\theta) + p(\theta+\pi)$ apart ($w(\theta)$ is called the *width* of $\boldsymbol{\gamma}$ in the direction $\theta$).
(vi) $\boldsymbol{\gamma}$ has a circumscribed square, i.e. a square all of whose sides are tangent to $\boldsymbol{\gamma}$.
(vii) If $\boldsymbol{\gamma}$ has *constant* width $D$, its length is $\pi D$;
(viii) Taking $p(\theta) = a\cos^2(k\theta/2) + b$, where $k$ is an odd integer and $a$ and $b$ are constants with $b > 0$, $a + b > 0$, gives a curve of constant width $a + 2b$.
(ix) The curve in (viii) is a circle if $|k| = 1$ but not if $|k| > 1$.

2.2.23 Show that if the parabola $y = \frac{1}{2}x^2$ rolls without slipping on the $x$-axis, the curved traced out by the point fixed relative to the parabola and initially at $(0,1)$ can be parametrized by

$$\boldsymbol{\gamma}(t) = \frac{1}{2}(t + \tanh t, \cosh t + \operatorname{sech} t).$$

2.2.24 Show that, if $\boldsymbol{\gamma}(t)$ is a closed curve of period $T_0$, and $\mathbf{t}$, $\mathbf{n}_s$ and $\kappa_s$ are its unit tangent vector, signed unit normal and signed curvature, respectively, then

$$\mathbf{t}(t+T_0) = \mathbf{t}(t), \quad \mathbf{n}_s(t+T_0) = \mathbf{n}_s(t), \quad \kappa_s(t+T_0) = \kappa_s(t).$$

2.3.1 Compute $\kappa, \tau, \mathbf{t}, \mathbf{n}$ and $\mathbf{b}$ for each of the following curves, and verify that the Frenet–Serret equations are satisfied:
(i) $\boldsymbol{\gamma}(t) = \left(\frac{1}{3}(1+t)^{3/2}, \frac{1}{3}(1-t)^{3/2}, \frac{t}{\sqrt{2}}\right)$.
(ii) $\boldsymbol{\gamma}(t) = \left(\frac{4}{5}\cos t, 1 - \sin t, -\frac{3}{5}\cos t\right)$.
Show that the curve in (ii) is a circle, and find its centre, radius and the plane in which it lies.

2.3.2 Describe all curves in $\mathbb{R}^3$ which have *constant* curvature $\kappa > 0$ and *constant* torsion $\tau$.

2.3.3 A regular curve $\boldsymbol{\gamma}$ in $\mathbb{R}^3$ with curvature $> 0$ is called a *generalized helix* if its tangent vector makes a fixed angle $\theta$ with a fixed unit vector $\mathbf{a}$. Show that the

torsion $\tau$ and curvature $\kappa$ of $\boldsymbol{\gamma}$ are related by $\tau = \pm\kappa\cot\theta$. Show conversely that, if the torsion and curvature of a regular curve are related by $\tau = \lambda\kappa$ where $\lambda$ is a constant, then the curve is a generalized helix.
In view of this result, Examples 2.1.3 and 2.3.2 show that a circular helix is a generalized helix. Verify this directly.

2.3.4 Let $\boldsymbol{\gamma}(t)$ be a unit-speed curve with $\kappa(t) > 0$ and $\tau(t) \neq 0$ for all $t$. Show that, if $\boldsymbol{\gamma}$ is *spherical*, i.e. if it lies on the surface of a sphere, then

$$\frac{\tau}{\kappa} = \frac{d}{ds}\left(\frac{\dot\kappa}{\tau\kappa^2}\right). \tag{2.22}$$

Conversely, show that if Eq. 2.22 holds, then

$$\rho^2 + (\dot\rho\sigma)^2 = r^2$$

for some (positive) constant $r$, where $\rho = 1/\kappa$ and $\sigma = 1/\tau$, and deduce that $\boldsymbol{\gamma}$ lies on a sphere of radius $r$. Verify that Eq. 2.22 holds for Viviani's curve (Exercise 1.1.8).

2.3.5 Let $P$ be an $n \times n$ orthogonal matrix and let $\mathbf{a} \in \mathbb{R}^n$, so that $M(\mathbf{v}) = P\mathbf{v} + \mathbf{a}$ is an isometry of $\mathbb{R}^3$ (see Appendix 1). Show that, if $\boldsymbol{\gamma}$ is a unit-speed curve in $\mathbb{R}^n$, the curve $\boldsymbol{\Gamma} = M(\boldsymbol{\gamma})$ is also unit-speed. Show also that, if $\mathbf{t}, \mathbf{n}, \mathbf{b}$ and $\mathbf{T}, \mathbf{N}, \mathbf{B}$ are the tangent vector, principal normal and binormal of $\boldsymbol{\gamma}$ and $\boldsymbol{\Gamma}$, respectively, then $\mathbf{T} = P\mathbf{t}$, $\mathbf{N} = P\mathbf{n}$ and $\mathbf{B} = P\mathbf{b}$.

2.3.6 Let $(a_{ij})$ be a skew-symmetric $3\times 3$ matrix (i.e. $a_{ij} = -a_{ji}$ for all $i, j$). Let $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$ be smooth functions of a parameter $s$ satisfying the differential equations

$$\dot{\mathbf{v}}_i = \sum_{j=1}^{3} a_{ij}\mathbf{v}_j,$$

for $i = 1, 2$ and 3, and suppose that for some parameter value $s_0$ the vectors $\mathbf{v}_1(s_0), \mathbf{v}_2(s_0)$ and $\mathbf{v}_3(s_0)$ are orthonormal. Show that the vectors $\mathbf{v}_1(s), \mathbf{v}_2(s)$ and $\mathbf{v}_3(s)$ are orthonormal for all values of $s$.

2.3.7 Repeat Exercise 2.3.1 for the following unit-speed curves:
(i) $\boldsymbol{\gamma}(t) = \left(\sin^2\frac{t}{\sqrt{2}}, \frac{1}{2}\sin t\sqrt{2}, \frac{t}{\sqrt{2}}\right)$.
(ii) $\boldsymbol{\gamma}(t) = \left(\frac{1}{\sqrt{3}}\cos t + \frac{1}{\sqrt{2}}\sin t, \frac{1}{\sqrt{3}}\cos t, \frac{1}{\sqrt{3}}\cos t - \frac{1}{\sqrt{2}}\sin t\right)$.

2.3.8 Repeat Exercise 2.3.1 for the curve

$$\boldsymbol{\gamma}(t) = \frac{1}{\sqrt{2}}(\cosh t, \sinh t, t)$$

(which is not unit-speed).

2.3.9 Show that the curve

$$\boldsymbol{\gamma}(t) = \left(\frac{1+t^2}{t}, t+1, \frac{1-t}{t}\right)$$

is planar.

2.3.10 Show that the curvature of the curve

$$\boldsymbol{\gamma}(t) = (t\cos(\ln t), t\sin(\ln t), t), \qquad t > 0$$

is proportional to $1/t$.

2.3.11 Show that the torsion of a regular curve $\boldsymbol{\gamma}(t)$ is a smooth function of $t$ whenever it is defined.

2.3.12 Let $\boldsymbol{\gamma}(t)$ be a unit-speed curve in $\mathbb{R}^3$, and assume that its curvature $\kappa(t)$ is non-zero for all $t$. Define a new curve $\boldsymbol{\delta}$ by

$$\boldsymbol{\delta}(t) = \frac{d\boldsymbol{\gamma}(t)}{dt}.$$

Show that $\boldsymbol{\delta}$ is regular and that, if $s$ is an arc-length parameter for $\boldsymbol{\delta}$, then

$$\frac{ds}{dt} = \kappa.$$

Prove that the curvature of $\boldsymbol{\delta}$ is

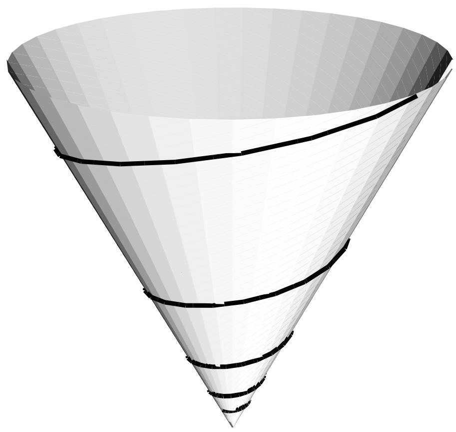
$$\left(1 + \frac{\tau^2}{\kappa^2}\right)^{\frac{1}{2}},$$

and find a formula for the torsion of $\boldsymbol{\delta}$ in terms of $\kappa$, $\tau$ and their derivatives with respect to $t$.

2.3.13 Show that the curve (shown below) on the cone

$$\boldsymbol{\sigma}(u, v) = (u\cos v, u\sin v, u)$$

given by $u = e^{\lambda t}$, $v = t$, where $\lambda$ is a constant, is a generalized helix.



2.3.14 Show that the twisted cubic

$$\boldsymbol{\gamma}(t) = (at, bt^2, ct^3),$$

where $a, b$ and $c$ are constants, is a generalized helix if and only if

$$3ac = \pm 2b^2.$$

2.3.15 A space curve $\tilde{\boldsymbol{\gamma}}$ is called a *Bertrand mate* of a space curve $\boldsymbol{\gamma}$ if, for each point $P$ of $\boldsymbol{\gamma}$, there is a point $\tilde{P}$ of $\tilde{\boldsymbol{\gamma}}$ such that the line $P\tilde{P}$ is parallel both to the principal normal of $\boldsymbol{\gamma}$ at $P$ and to the principal normal of $\tilde{\boldsymbol{\gamma}}$ at $\tilde{P}$. If $\boldsymbol{\gamma}$ has a Bertrand mate it is called a *Bertrand curve*.
Assume that $\boldsymbol{\gamma}$ and $\tilde{\boldsymbol{\gamma}}$ are unit-speed and let $\tilde{\boldsymbol{\gamma}}(\tilde{s})$ be the point of $\tilde{\boldsymbol{\gamma}}$ corresponding to the point $\boldsymbol{\gamma}(s)$ of $\boldsymbol{\gamma}$, where $\tilde{s}$ is a smooth function of $s$. Show that:
(i) $\tilde{\boldsymbol{\gamma}}(\tilde{s}) = \boldsymbol{\gamma}(s) + a\mathbf{n}(s)$, where $\mathbf{n}$ is the principal normal of $\boldsymbol{\gamma}$ and $a$ is a *constant*.
(ii) There is a *constant* $\alpha$ such that the tangent vector, principal normal and binormal of $\boldsymbol{\gamma}$ and $\tilde{\boldsymbol{\gamma}}$ at corresponding points are related by

$$\tilde{\mathbf{t}} = \cos\alpha\mathbf{t} - \sin\alpha\mathbf{b}, \quad \tilde{\mathbf{n}} = \pm\mathbf{n}, \quad \tilde{\mathbf{b}} = \pm(\sin\alpha\mathbf{t} + \cos\alpha\mathbf{b}),$$

where the signs in the last two equations are the same.
(iii) The curvature and torsion of $\boldsymbol{\gamma}$ and $\tilde{\boldsymbol{\gamma}}$ at corresponding points are related by

$$\cos\alpha\frac{d\tilde{s}}{ds} = 1 - a\kappa, \qquad \sin\alpha\frac{d\tilde{s}}{ds} = -a\tau,$$
$$\cos\alpha\frac{ds}{d\tilde{s}} = 1 + a\tilde{\kappa}, \qquad \sin\alpha\frac{ds}{d\tilde{s}} = -a\tilde{\tau}.$$

(iv) $a\kappa - a\tau\cot\alpha = 1$.
(v) $a^2\tau\tilde{\tau} = \sin^2\alpha$, $(1 - a\kappa)(1 + a\tilde{\kappa}) = \cos^2\alpha$.

2.3.16 Show that every plane curve is a Bertrand curve.

2.3.17 Show that a space curve $\boldsymbol{\gamma}$ with nowhere vanishing curvature $\kappa$ and nowhere vanishing torsion $\tau$ is a Bertrand curve if and only if there exist constants $a, b$ such that

$$a\kappa + b\tau = 1.$$

2.3.18 Show that a Bertrand curve $\mathcal{C}$ with nowhere vanishing curvature and torsion has more than one Bertrand mate if and only if it is a circular helix, in which case it has infinitely-many Bertrand mates, all of which are circular helices with the same axis and pitch as $\mathcal{C}$.

2.3.19 Show that a spherical curve of constant curvature is a circle.

2.3.20 The *normal plane* at a point $P$ of a space curve $\mathcal{C}$ is the plane passing through $P$ perpendicular to the tangent line of $\mathcal{C}$ at $P$. Show that, if all the normal planes of a curve pass through some fixed point, the curve is spherical.

2.3.21 Let $\boldsymbol{\gamma}$ be a curve in $\mathbb{R}^3$ and let $\Pi$ be a plane

$$\mathbf{v}.\mathbf{N} = d,$$

where $\mathbf{N}$ and $d$ are constants with $\mathbf{N} \neq \mathbf{0}$, and $\mathbf{v} = (x, y, z)$. Let

$$F(t) = \boldsymbol{\gamma}(t).\mathbf{N} - d.$$

Show that:
) $F(t_0) = 0$ if and only if $\boldsymbol{\gamma}$ intersects $\Pi$ at the point $\boldsymbol{\gamma}(t_0)$.
(ii) $F(t_0) = \dot{F}(t_0) = 0$ if and only if $\boldsymbol{\gamma}$ touches $\Pi$ at $\boldsymbol{\gamma}(t_0)$ (i.e. $\dot{\boldsymbol{\gamma}}(t_0)$ is parallel to $\Pi$).
(iii) If the curvature of $\boldsymbol{\gamma}$ at $\boldsymbol{\gamma}(t_0)$ is non-zero, there is a unique plane $\Pi$ such that

$$F(t_0) = \dot{F}(t_0) = \ddot{F}(t_0) = 0,$$

and that this plane $\Pi$ is the plane passing through $\boldsymbol{\gamma}(t_0)$ parallel to $\dot{\boldsymbol{\gamma}}(t_0)$ and $\ddot{\boldsymbol{\gamma}}(t_0)$ ($\Pi$ is called the *osculating plane* of $\boldsymbol{\gamma}$ at $\boldsymbol{\gamma}(t_0)$; intuitively, it is the plane which most closely approaches $\boldsymbol{\gamma}$ near the point $\boldsymbol{\gamma}(t_0)$).
(iv) If $\boldsymbol{\gamma}$ is contained in a plane $\Pi'$, then $\Pi'$ is the osculating plane of $\boldsymbol{\gamma}$ at each of its points.
(v) If the torsion of $\boldsymbol{\gamma}$ is non-zero at $\boldsymbol{\gamma}(t_0)$, then $\boldsymbol{\gamma}$ crosses its osculating plane there.
Compare Exercise 2.2.17.

2.3.22 Find the osculating plane at a general point of the circular helix

$$\boldsymbol{\gamma}(t) = (a\cos t, a\sin t, bt).$$

2.3.23 Show that the osculating planes at any three distinct points $P_1, P_2, P_3$ of the twisted cubic

$$\boldsymbol{\gamma}(t) = (t, t^2, t^3)$$

meet at a single point $Q$, and that the four points $P_1, P_2, P_3, Q$ all lie in a plane.

2.3.24 Suppose that a curve $\boldsymbol{\gamma}$ has nowhere vanishing curvature and that each of its osculating planes pass through some fixed point. Prove that the curve lies in a plane.

2.3.25 Show that the orthogonal projection of a curve $\mathcal{C}$ onto its normal plane at a point $P$ of $\mathcal{C}$ is a plane curve which has an ordinary cusp at $P$ provided that $\mathcal{C}$ has non-zero curvature and torsion at $P$ (see Exercise 1.3.3). Show, on the other hand, that $P$ is a regular point of the orthogonal projection of $\mathcal{C}$ onto its osculating plane at $P$.

2.3.26 Let $\mathcal{S}$ be the sphere with centre $\mathbf{c}$ and radius $R$. Let $\boldsymbol{\gamma}$ be a unit-speed curve in $\mathbb{R}^3$ and let

$$F(t) = \parallel \boldsymbol{\gamma}(t) - \mathbf{c} \parallel^2 - R^2.$$

Let $t_0 \in \mathbb{R}$. Show that:
(i) $F(t_0) = 0$ if and only if $\boldsymbol{\gamma}$ intersects $\mathcal{S}$ at the point $\boldsymbol{\gamma}(t_0)$.
(ii) $F(t_0) = \dot{F}(t_0) = 0$ if and only if $\boldsymbol{\gamma}$ is tangent to $\mathcal{S}$ at $\boldsymbol{\gamma}(t_0)$.
Compute $\ddot{F}$ and $\dddot{F}$ and show that there is a unique sphere $\mathcal{S}$ (called the *osculating sphere* of $\boldsymbol{\gamma}$ at $\boldsymbol{\gamma}(t_0)$) such that

$$F(t_0) = \dot{F}(t_0) = \ddot{F}(t_0) = \dddot{F}(t_0) = 0.$$

Show that the centre of $\mathcal{S}$ is

$$\mathbf{c} = \boldsymbol{\gamma} + \frac{1}{\kappa}\mathbf{n} - \frac{\dot{\kappa}}{\kappa^2\tau}\mathbf{b}$$

in the usual notation, all quantities being evaluated at $t = t_0$. What is its radius? The point $\mathbf{c}(t_0)$ is called the *centre of spherical curvature* of $\boldsymbol{\gamma}$ at $\boldsymbol{\gamma}(t_0)$. Show that $\mathbf{c}(t_0)$ is independent of $t_0$ if and only if $\boldsymbol{\gamma}$ is spherical, in which case the sphere on which $\boldsymbol{\gamma}$ lies is its osculating sphere.

2.3.27 The *osculating circle* of a curve $\boldsymbol{\gamma}$ at a point $\boldsymbol{\gamma}(t_0)$ is the intersection of the osculating plane and the osculating sphere of $\boldsymbol{\gamma}$ at $\boldsymbol{\gamma}(t_0)$. Show that the centre of the osculating circle is the *centre of curvature*

$$\boldsymbol{\gamma} + \frac{1}{\kappa}\mathbf{n},$$

and that its radius is $1/\kappa$, all quantities being evaluated at $t = t_0$. (Compare Exercise 2.2.17.)

# Chapter 3

3.1.1 Show that

$$\boldsymbol{\gamma}(t) = ((1 + a\cos t)\cos t, (1 + a\cos t)\sin t),$$

where $a$ is a constant, is a simple closed curve if $|a| < 1$, but that if $|a| > 1$ its complement is the disjoint union of three connected subsets of $\mathbb{R}^2$, two of which are bounded and one is unbounded. What happens if $a = \pm 1$?

3.1.2 Show that, if $\boldsymbol{\gamma}$ is as in Exercise 3.1.1, its turning angle $\varphi$ satisfies

$$\frac{d\varphi}{dt} = 1 + \frac{a(\cos t + a)}{1 + 2a\cos t + a^2}.$$

Deduce that
$$\int_0^{2\pi} \frac{a(\cos t + a)}{1 + 2a\cos t + a^2}\,dt = \begin{cases} 0 & \text{if } |a| < 1, \\ 2\pi & \text{if } |a| > 1. \end{cases}$$

3.2.1 Show that the length $\ell(\boldsymbol{\gamma})$ and the area $\mathcal{A}(\boldsymbol{\gamma})$ are unchanged by applying an isometry to $\boldsymbol{\gamma}$.

3.2.2 By applying the isoperimetric inequality to the ellipse
$$\frac{x^2}{p^2} + \frac{y^2}{q^2} = 1$$
(where $p$ and $q$ are positive constants), prove that
$$\int_0^{2\pi} \sqrt{p^2 \sin^2 t + q^2 \cos^2 t}\,dt \geq 2\pi\sqrt{pq},$$
with equality holding if and only if $p = q$.

3.2.3 What is the area of the interior of the ellipse
$$\boldsymbol{\gamma}(t) = (p\cos t, q\sin t),$$
where $p$ and $q$ are positive constants?

3.3.1 Show that the ellipse in Example 3.1.2 is convex.

3.3.2 Show that the limaçn in Example 1.1.7 has only two vertices (cf. Example 3.1.3).

3.3.3 Show that a plane curve $\boldsymbol{\gamma}$ has a vertex at $t = t_0$ if and only if the evolute $\boldsymbol{\epsilon}$ of $\boldsymbol{\gamma}$ (Exercise 2.2.7) has a singular point at $t = t_0$.

3.3.4 Show that the vertices of the curve $y = f(x)$ satisfy
$$\left(1 + \left(\frac{df}{dx}\right)^2\right)\frac{d^3 f}{dx^3} = 3\frac{df}{dx}\left(\frac{d^2 f}{dx^2}\right)^2.$$

3.3.5 Show that the curve
$$\boldsymbol{\gamma}(t) = (at - b\sin t, a - b\cos t),$$
where $a$ and $b$ are non-zero constants, has vertices at the points $\boldsymbol{\gamma}(n\pi)$ for all integers $n$. Show that these are all the vertices of $\boldsymbol{\gamma}$ unless
$$\frac{a-b}{b} \leq \frac{2b}{a} \leq \frac{a+b}{b},$$

in which case there are infinitely-many other vertices.

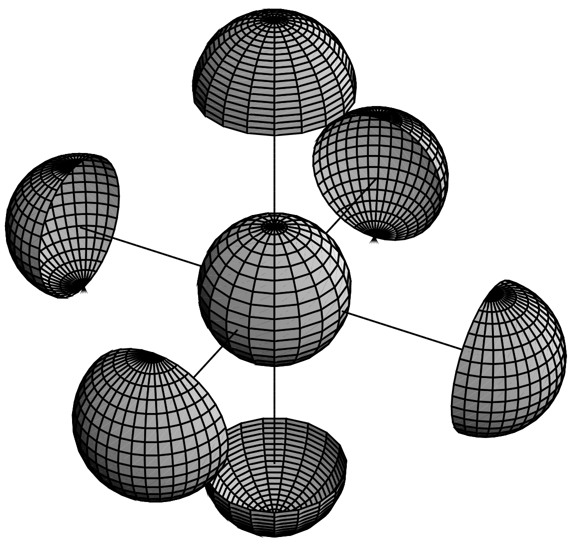
# Chapter 4

4.1.1 Show that any open disc in the $xy$-plane is a surface.

4.1.2 Define surface patches $\boldsymbol{\sigma}^x_\pm : U \to \mathbb{R}^3$ for $S^2$ by solving the equation $x^2+y^2+z^2=1$ for $x$ in terms of $y$ and $z$:

$$\boldsymbol{\sigma}^x_\pm(u,v) = (\pm\sqrt{1-u^2-v^2}, u, v),$$

defined on the open set $U = \{(u,v) \in \mathbb{R}^2 \mid u^2+v^2 < 1\}$. Define $\boldsymbol{\sigma}^y_\pm$ and $\boldsymbol{\sigma}^z_\pm$ similarly (with the same $U$) by solving for $y$ and $z$, respectively. Show that these six patches give $S^2$ the structure of a surface.



4.1.3 The *hyperboloid of one sheet* is

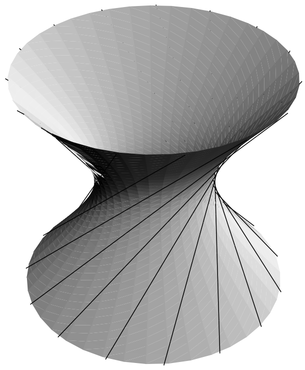
$$\mathcal{S} = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2-z^2=1\}.$$

Show that, for every $\theta$, the straight line

$$(x-z)\cos\theta = (1-y)\sin\theta, \quad (x+z)\sin\theta = (1+y)\cos\theta$$

is contained in $\mathcal{S}$, and that every point of the hyperboloid lies on one of these lines. Deduce that $\mathcal{S}$ can be covered by a single surface patch, and hence is a surface. (Compare the case of the cylinder in Example 4.1.3.)

Find a second family of straight lines on $\mathcal{S}$, and show that no two lines of the same family intersect, while every line of the first family intersects every line of the second family with one exception. One says that the surface $\mathcal{S}$ is *doubly ruled.*



4.1.4 Show that the unit cylinder can be covered by a single surface patch, but that the unit sphere cannot. (The second part requires some point set topology.)

4.1.5 Show that every open subset of a surface is a surface.

4.1.6 Show that a curve on the unit cylinder that intersects the straight lines on the cylinder parallel to the $z$-axis at a constant angle must be a straight line, a circle or a circular helix.

4.1.7 Find a surface patch for the *ellipsoid*

$$\frac{x^2}{p^2} + \frac{y^2}{q^2} + \frac{z^2}{r^2} = 1,$$

where $p, q$ and $r$ are non-zero constants. (A picture of an ellipsoid can be found in Theorem 5.2.2.)

4.1.8 Show that

$$\boldsymbol{\sigma}(u, v) = (\sin u, \sin v, \sin(u + v)), \quad -\pi/2 < u, v < \pi/2$$

is a surface patch for the surface with Cartesian equation

$$(x^2 - y^2 + z^2)^2 = 4x^2 z^2 (1 - y^2).$$

4.2.1 Show that, if $f(x, y)$ is a smooth function, its *graph*

$$\{(x, y, z) \in \mathbb{R}^3 \mid z = f(x, y)\}$$

is a smooth surface with atlas consisting of the single regular surface patch

$$\boldsymbol{\sigma}(u,v) = (u, v, f(u,v)).$$

In fact, every surface is "locally" of this type - see Exercise 5.6.4.

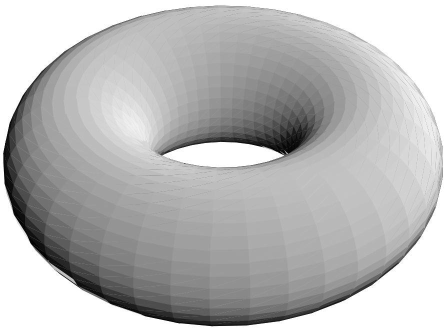
4.2.2 Verify that the six surface patches for $S^2$ in Exercise 4.1.2 are regular. Calculate the transition maps between them and verify that these maps are smooth.

4.2.3 Which of the following are regular surface patches (in each case, $u, v \in \mathbb{R}$):
(i) $\boldsymbol{\sigma}(u,v) = (u, v, uv)$.
(ii) $\boldsymbol{\sigma}(u,v) = (u, v^2, v^3)$.
(iii) $\boldsymbol{\sigma}(u,v) = (u + u^2, v, v^2)$ ?

4.2.4 Show that the ellipsoid

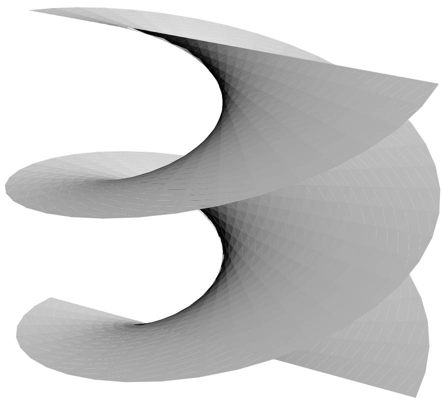
$$\frac{x^2}{p^2} + \frac{y^2}{q^2} + \frac{z^2}{r^2} = 1,$$

where $p$, $q$ and $r$ are non-zero constants, is a smooth surface.



4.2.5 A *torus* (see above) is obtained by rotating a circle $\mathcal{C}$ in a plane $\Pi$ around a straight line $l$ in $\Pi$ that does not intersect $\mathcal{C}$. Take $\Pi$ to be the $xz$-plane, $l$ to be the $z$-axis, $a > 0$ the distance of the centre of $\mathcal{C}$ from $l$, and $b < a$ the radius of $\mathcal{C}$. Show that the torus is a smooth surface with parametrization

$$\boldsymbol{\sigma}(\theta, \varphi) = ((a + b\cos\theta)\cos\varphi, (a + b\cos\theta)\sin\varphi, b\sin\theta).$$



4.2.6 A *helicoid* is the surface swept out by an aeroplane propeller, when both the aeroplane and its propeller move at constant speed (see the picture above). If the aeroplane is flying along the $z$-axis, show that the helicoid can be parametrized as

$$\boldsymbol{\sigma}(u, v) = (v \cos u, v \sin u, \lambda u),$$

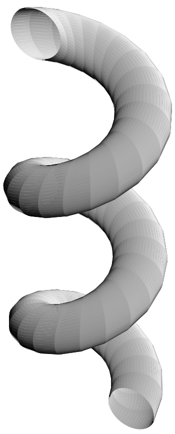
where $\lambda$ is a constant. Show that the cotangent of the angle that the standard unit normal of $\boldsymbol{\sigma}$ at a point $\mathbf{p}$ makes with the $z$-axis is proportional to the distance of $\mathbf{p}$ from the $z$-axis.

4.2.7 Let $\boldsymbol{\gamma}$ be a unit-speed curve in $\mathbb{R}^3$ with nowhere vanishing curvature. The *tube* of radius $a > 0$ around $\boldsymbol{\gamma}$ is the surface parametrized by

$$\boldsymbol{\sigma}(s, \theta) = \boldsymbol{\gamma}(s) + a(\mathbf{n}(s) \cos \theta + \mathbf{b}(s) \sin \theta),$$

where $\mathbf{n}$ is the principal normal of $\boldsymbol{\gamma}$ and $\mathbf{b}$ is its binormal. Give a geometrical description of this surface. Prove that $\boldsymbol{\sigma}$ is regular if the curvature $\kappa$ of $\boldsymbol{\gamma}$ is less than $a^{-1}$ everywhere.

Note that, even if $\boldsymbol{\sigma}$ is regular, the surface $\boldsymbol{\sigma}$ will have self-intersections if the curve $\boldsymbol{\gamma}$ comes within a distance $2a$ of itself. This illustrates the fact that regularity is a *local* property: if $(s, \theta)$ is restricted to lie in a sufficiently small open subset $U$ of $\mathbb{R}^2$, $\boldsymbol{\sigma} : U \to \mathbb{R}^3$ will be smooth and injective (so there will be no self-intersections) - see Exercise 5.6.3. We shall see other instances of this later (e.g. Example 12.2.5).



The tube around a circular helix

4.2.8 Show that translations and invertible linear transformations of $\mathbb{R}^3$ take smooth surfaces to smooth surfaces.

4.2.9 Show that every open subset of a smooth surface is a smooth surface.

4.2.10 Show that the graph in Exercise 4.2.1 is diffeomorphic to an open subset of a plane.

4.2.11 Show that the surface patch in Exercise 4.1.8 is regular.

4.2.12 Show that the torus in Exercise 4.2.5 can be covered by *three* patches $\boldsymbol{\sigma}(\theta, \varphi)$, with $(\theta, \varphi)$ lying in an open rectangle in $\mathbb{R}^2$, but not by two.

4.2.13 For which values of the constant $c$ is

$$z(z+4) = 3xy + c$$

a smooth surface?

4.2.14 Show that

$$x^3 + 3(y^2 + z^2)^2 = 2$$

is a smooth surface.

4.2.15 Let $\mathcal{S}$ be the *astroidal sphere*

$$x^{2/3} + y^{2/3} + z^{2/3} = 1.$$

Show that, if we exclude from $\mathcal{S}$ its intersections with the coordinate planes, we obtain a smooth surface $\tilde{\mathcal{S}}$.

4.2.16 Show that the surface

$$xyz = 1$$

is not connected, but that it is the disjoint union of four connected surfaces. Find a parametrization of each connected piece.

4.2.17 Show that the set of mid-points of the chords of a circular helix is a subset of a helicoid.

4.3.1 If $\mathcal{S}$ is a smooth surface, define the notion of a *smooth function* $\mathcal{S} \to \mathbb{R}$. Show that, if $\mathcal{S}$ is a smooth surface, each component of the inclusion map $\mathcal{S} \to \mathbb{R}^3$ is a smooth function $\mathcal{S} \to \mathbb{R}$.

4.3.2 Let $\mathcal{S}$ be the half-cone $x^2 + y^2 = z^2$, $z > 0$ (see Example 4.1.5). Define a map $f$ from the half-plane $\{(0, y, z) \,|\, y > 0\}$ to $\mathcal{S}$ by $f(0, y, z) = (y \cos z, y \sin z, y)$. Show that $f$ is a local diffeomorphism but not a diffeomorphism.

4.4.1 Find the equation of the tangent plane of each of the following surface patches at the indicated points:
(i) $\boldsymbol{\sigma}(u, v) = (u, v, u^2 - v^2)$, $(1, 1, 0)$.
(ii) $\boldsymbol{\sigma}(r, \theta) = (r \cosh \theta, r \sinh \theta, r^2)$, $(1, 0, 1)$.

4.4.2 Show that, if $\boldsymbol{\sigma}(u, v)$ is a surface patch, the set of linear combinations of $\boldsymbol{\sigma}_u$ and $\boldsymbol{\sigma}_v$ is unchanged when $\boldsymbol{\sigma}$ is reparametrized.

4.4.3 Let $\mathcal{S}$ be a surface, let $\mathbf{p} \in \mathcal{S}$ and let $F : \mathbb{R}^3 \to \mathbb{R}$ be a smooth function. Let $\boldsymbol{\nabla}_{\mathcal{S}} F$ be the perpendicular projection of the gradient $\boldsymbol{\nabla} F = (F_x, F_y, F_z)$ of $F$ onto $T_{\mathbf{p}}\mathcal{S}$. Show that, if $\boldsymbol{\gamma}$ is any curve on $\mathcal{S}$ passing through $\mathbf{p}$ when $t = t_0$, say,

$$(\boldsymbol{\nabla}_{\mathcal{S}} F) . \dot{\boldsymbol{\gamma}}(t_0) = \left.\frac{d}{dt}\right|_{t=t_0} F(\boldsymbol{\gamma}(t)).$$

Deduce that $\boldsymbol{\nabla}_{\mathcal{S}} F = \mathbf{0}$ if the restriction of $F$ to $\mathcal{S}$ has a local maximum or a local minimum at $\mathbf{p}$.

4.4.4 Let $f : \mathcal{S}_1 \to \mathcal{S}_2$ be a local diffeomorphism and let $\boldsymbol{\gamma}$ be a regular curve on $\mathcal{S}_1$. Show that $f \circ \boldsymbol{\gamma}$ is a regular curve on $\mathcal{S}_2$.

4.4.5 Find the equation of the tangent plane of the torus in Exercise 4.2.5 at the point corresponding to $\theta = \varphi = \pi/4$.

4.5.1 Calculate the transition map $\Phi$ between the two surface patches for the Möbius band in Example 4.5.3. Show that it is defined on the union of two disjoint rectangles in $\mathbb{R}^2$, and that the determinant of the Jacobian matrix of $\Phi$ is equal to $+1$ on one of the rectangles and to $-1$ on the other.

4.5.2 Suppose that two smooth surfaces $\mathcal{S}$ and $\tilde{\mathcal{S}}$ are diffeomorphic and that $\mathcal{S}$ is orientable. Prove that $\tilde{\mathcal{S}}$ is orientable.

4.5.3 Show that for the latitude-longitude parametrization of $S^2$ (Example 4.1.4) the standard unit normal points inwards. What about the parametrizations given in Exercise 4.1.2?

4.5.4 Let $\boldsymbol{\gamma}$ be a curve on a surface patch $\boldsymbol{\sigma}$, and let $\mathbf{v}$ be a unit *vector field along* $\boldsymbol{\gamma}$, i.e. $\mathbf{v}(t)$ is a unit tangent vector to $\boldsymbol{\sigma}$ for all values of the curve parameter $t$, and $\mathbf{v}$ is a smooth function of $t$. Let $\tilde{\mathbf{v}}$ be the result of applying a positive rotation through $\pi/2$ to $\mathbf{v}$. Suppose that, for some fixed parameter value $t_0$,

$$\dot{\boldsymbol{\gamma}}(t_0) = \cos\theta_0 \mathbf{v}(t_0) + \sin\theta_0 \tilde{\mathbf{v}}(t_0).$$

Show that there is a *smooth* function $\theta(t)$ such that $\theta(t_0) = \theta_0$ and

$$\dot{\boldsymbol{\gamma}}(t) = \cos\theta(t)\mathbf{v}(t) + \sin\theta(t)\tilde{\mathbf{v}}(t) \quad \text{for all } t.$$

4.5.5 The map $F : \mathbb{R}^3 \backslash \{(0,0,0)\} \to \mathbb{R}^3 \backslash \{(0,0,0)\}$ given by

$$F(\mathbf{v}) = \frac{\mathbf{v}}{\mathbf{v}.\mathbf{v}}$$

is called *inversion* with respect to $S^2$ (compare the discussion of inversion in circles in Appendix 2). Geometrically, $F(\mathbf{v})$ is the point on the radius from the origin passing through $\mathbf{v}$ such that the product of the distances of $\mathbf{v}$ and $F(\mathbf{v})$

from the origin is equal to 1. Let $\mathcal{S}$ be a surface that does not pass through the origin, and let $\mathcal{S}^* = F(\mathcal{S})$. Show that, if $\mathcal{S}$ is orientable, then so is $\mathcal{S}^*$. Show, in fact, that if $\mathbf{N}$ is the unit normal of $\mathcal{S}$ at a point $\mathbf{p}$, that of $\mathcal{S}^*$ at $F(\mathbf{p})$ is

$$\mathbf{N}^* = \frac{2(\mathbf{p}.\mathbf{N})}{\parallel \mathbf{p} \parallel^2}\mathbf{p} - \mathbf{N}.$$

# Chapter 5

5.1.1 Show that the following are smooth surfaces:
(i) $x^2 + y^2 + z^4 = 1$;
(ii) $(x^2 + y^2 + z^2 + a^2 - b^2)^2 = 4a^2(x^2 + y^2)$, where $a > b > 0$ are constants.
Show that the surface in (ii) is, in fact, the torus of Exercise 4.2.5.

5.1.2 Consider the surface $\mathcal{S}$ defined by $f(x, y, z) = 0$, where $f$ is a smooth function such that $\nabla f$ does not vanish at any point of $\mathcal{S}$. Show that $\nabla f$ is perpendicular to the tangent plane at every point of $\mathcal{S}$, and deduce that $\mathcal{S}$ is orientable.
Suppose now that $F : \mathbb{R}^3 \to \mathbb{R}$ is a smooth function. Show that, if the restriction of $F$ to $\mathcal{S}$ has a local maximum or a local minimum at $\mathbf{p}$ then, at $\mathbf{p}$, $\nabla F = \lambda \nabla f$ for some scalar $\lambda$. (This is called *Lagrange's Method of Undetermined Multipliers.*)

5.1.3 Show that the smallest value of $x^2 + y^2 + z^2$ subject to the condition $xyz = 1$ is 3, and that the points $(x, y, z)$ that give this minimum value lie at the vertices of a regular tetrahedron in $\mathbb{R}^3$.

5.2.1 Write down parametrizations of each of the quadrics in parts (i)–(xi) of Theorem 5.2.2 (in case (vi) one must remove the origin).

5.2.2 Show that the quadric

$$x^2 + y^2 - 2z^2 - \frac{2}{3}xy + 4z = c$$

is a hyperboloid of one sheet if $c > 2$, and a hyperboloid of two sheets if $c < 2$. What if $c = 2$? (This exercise requires a knowledge of eigenvalues and eigenvectors.)

5.2.3 Show that, if a quadric contains three points on a straight line, it contains the whole line. Deduce that, if $L_1, L_2$ and $L_3$ are non-intersecting straight lines in $\mathbb{R}^3$, there is a quadric containing all three lines.

5.2.4 Use the preceding exercise to show that any doubly ruled surface is (part of) a quadric surface. (A surface is doubly ruled if it is the union of each of two families of straight lines such that no two lines of the same family intersect, but every line of the first family intersects every line of the second family, with at most a finite number of exceptions.) Which quadric surfaces are doubly ruled?

5.2.5 By setting

$$u = \frac{x}{p} - \frac{y}{q}, \qquad v = \frac{x}{p} + \frac{y}{q},$$

find a surface patch covering the hyperbolic paraboloid

$$\frac{x^2}{p^2} - \frac{y^2}{q^2} = z.$$

Deduce that the hyperbolic paraboloid is doubly ruled.

5.2.6 A *conic* is a level curve of the form

$$ax^2 + by^2 + 2cxy + dx + ey + f = 0,$$

where the coefficients $a, b, c, d, e$ and $f$ are constants, not all of which are zero. By imitating the proof of Theorem 5.2.2, show that any non-empty conic that is not a straight line or a single point can be transformed by applying a direct isometry of $\mathbb{R}^2$ into one of the following:

(i) An ellipse $\frac{x^2}{p^2} + \frac{y^2}{q^2} = 1$.
(ii) A parabola $y^2 = 2px$.
(iii) A hyperbola $\frac{x^2}{p^2} - \frac{y^2}{q^2} = 1$.
(iv) A pair of intersecting straight lines $y^2 = p^2x^2$.

Here, $p$ and $q$ are non-zero constants.

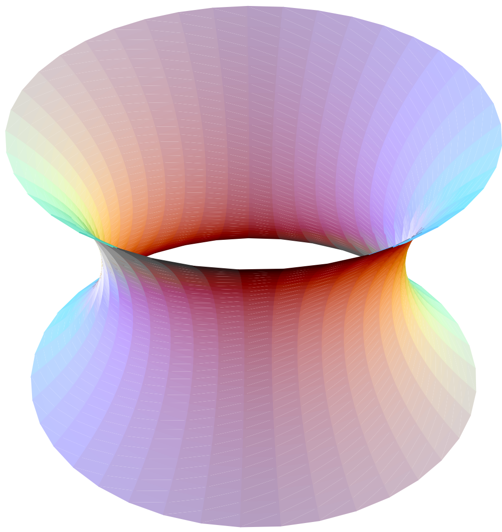
5.2.7 Show that:
(i) Any connected quadric surface is diffeomorphic to a sphere, a circular cylinder or a plane.
(ii) Each connected piece of a non-connected quadric surface is diffeomorphic to a plane.

5.3.1 The surface obtained by rotating the curve $x = \cosh z$ in the $xz$-plane around the $z$-axis is called a *catenoid* (illustrated below). Describe an atlas for this surface.

5.3.2 Show that

$$\boldsymbol{\sigma}(u, v) = (\operatorname{sech} u \cos v, \operatorname{sech} u \sin v, \tanh u)$$

is a regular surface patch for $S^2$ (it is called *Mercator's projection*). Show that meridians and parallels on $S^2$ correspond under $\boldsymbol{\sigma}$ to perpendicular straight lines in the plane. (This patch is 'derived' in Exercise 6.3.3.)



5.3.3 Show that, if $\boldsymbol{\sigma}(u,v)$ is the (generalized) cylinder in Example 5.3.1:
(i) The curve $\tilde{\boldsymbol{\gamma}}(u) = \boldsymbol{\gamma}(u) - (\boldsymbol{\gamma}(u).\mathbf{a})\mathbf{a}$ is contained in a plane perpendicular to $\mathbf{a}$.
(ii) $\boldsymbol{\sigma}(u,v) = \tilde{\boldsymbol{\gamma}}(u) + \tilde{v}\mathbf{a}$, where $\tilde{v} = v + \boldsymbol{\gamma}(u).\mathbf{a}$.
(iii) $\tilde{\boldsymbol{\sigma}}(u,\tilde{v}) = \tilde{\boldsymbol{\gamma}}(u) + \tilde{v}\mathbf{a}$ is a reparametrization of $\boldsymbol{\sigma}(u,v)$.
This exercise shows that, when considering a generalized cylinder $\boldsymbol{\sigma}(u,v) = \boldsymbol{\gamma}(u) + v\mathbf{a}$, we can always assume that the curve $\boldsymbol{\gamma}$ is contained in a plane perpendicular to the vector $\mathbf{a}$.

5.3.4 Consider the ruled surface

$$\boldsymbol{\sigma}(u,v) = \boldsymbol{\gamma}(u) + v\boldsymbol{\delta}(u), \tag{5.5}$$

where $\parallel \boldsymbol{\delta}(u) \parallel = 1$ and $\dot{\boldsymbol{\delta}}(u) \neq \mathbf{0}$ for all values of $u$ (a dot denotes $d/du$). Show that there is a unique point $\boldsymbol{\Gamma}(u)$, say, on the ruling through $\boldsymbol{\gamma}(u)$ at which $\dot{\boldsymbol{\delta}}(u)$ is perpendicular to the surface. The curve $\boldsymbol{\Gamma}$ is called the *line of striction* of the ruled surface $\boldsymbol{\sigma}$ (of course, it need not be a straight line). Show that $\dot{\boldsymbol{\Gamma}}.\dot{\boldsymbol{\delta}} = 0$.
Let $\tilde{v} = v + \frac{\dot{\boldsymbol{\gamma}}.\dot{\boldsymbol{\delta}}}{\|\dot{\boldsymbol{\delta}}\|^2}$, and let $\tilde{\boldsymbol{\sigma}}(u,\tilde{v})$ be the corresponding reparametrization of $\boldsymbol{\sigma}$. Then, $\tilde{\boldsymbol{\sigma}}(u,\tilde{v}) = \boldsymbol{\Gamma}(u) + \tilde{v}\boldsymbol{\delta}(u)$. This means that, when considering ruled surfaces as in (5.5), *we can always assume that* $\dot{\boldsymbol{\gamma}}.\dot{\boldsymbol{\delta}} = 0$. We shall make use of this in Chapter 12.

5.3.5 A *loxodrome* is a curve on a sphere that intersects the meridians at a fixed angle, say $\alpha$. Show that, in the Mercator surface patch $\boldsymbol{\sigma}$ of $S^2$ in Exercise 5.3.2, a unit-speed loxodrome satisfies

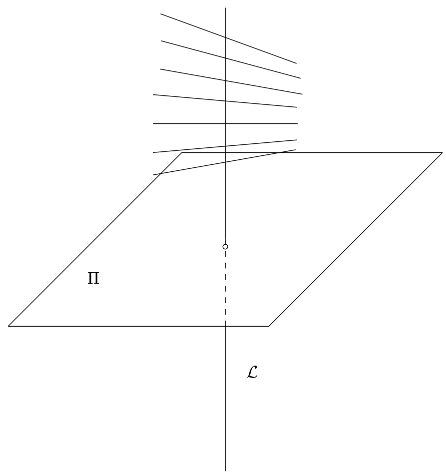
$$\dot{u} = \cos\alpha\cosh u, \quad \dot{v} = \pm\sin\alpha\cosh u$$

(a dot denoting differentiation with respect to the parameter of the loxodrome). Deduce that loxodromes correspond under $\boldsymbol{\sigma}$ to straight lines in the $uv$-plane.

5.3.6 A *conoid* is a ruled surface whose rulings are parallel to a given plane $\Pi$ and pass through a given straight line $\mathcal{L}$ perpendicular to $\Pi$. If $\Pi$ is the $xy$-plane and $\mathcal{L}$ is the $z$-axis, show that

$$\boldsymbol{\sigma}(u, \theta) = (u \cos\theta, u \sin\theta, f(\theta)), \qquad u \neq 0$$

is a regular surface patch for the conoid, where $\theta$ is the angle between a ruling and the positive $x$-axis and $f(\theta)$ is the height above $\Pi$ at which the ruling intersects $\mathcal{L}$ ($f$ is assumed to be smooth).




5.3.7 The *normal line* at a point $P$ of a surface $\boldsymbol{\sigma}$ is the straight line passing through $P$ parallel to the normal $\mathbf{N}$ of $\boldsymbol{\sigma}$ at $P$. Prove that:
(i) If the normal lines are all parallel, then $\boldsymbol{\sigma}$ is an open subset of a plane.
(ii) If all the normal lines pass through some fixed point, then $\boldsymbol{\sigma}$ is an open subset of a sphere.
(iii) If all the normal lines intersect a given straight line, then $\boldsymbol{\sigma}$ is an open subset of a surface of revolution.

5.3.8 Show that the line of striction of the hyperboloid of one sheet

$$x^2 + y^2 - z^2 = 1$$

is the circle in which the surface intersects the $xy$-plane (recall from Exercise 4.1.3 that this surface is ruled.)

5.3.9 Which quadric surfaces are:
(a) Generalized cylinders.
(b) Generalized cones.
(c) Ruled surfaces.

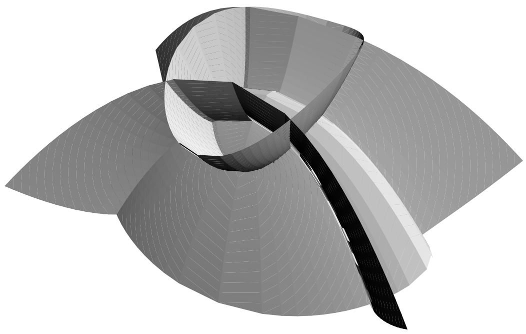
(d) Surfaces of revolution?

5.3.10 Let $\mathcal{S}$ be a ruled surface. Show that the union of the normal lines (Exercise 5.3.7) at the points of a ruling of $\mathcal{S}$ is a plane or a hyperbolic paraboloid.

5.4.1 One of the following surfaces is compact and one is not:
(i) $x^2 - y^2 + z^4 = 1$.
(ii) $x^2 + y^2 + z^4 = 1$.
Which is which, and why? Sketch the compact surface.

5.4.2 Explain, without giving a detailed proof, why the tube (Exercise 4.2.7) around a closed curve in $\mathbb{R}^3$ with no self-intersections is a compact surface diffeomorphic to a torus (provided the tube has sufficiently small radius).

5.5.1 Show that the following are triply orthogonal systems:
(i) The spheres with centre the origin, the planes containing the $z$-axis, and the circular cones with axis the $z$-axis.
(ii) The planes parallel to the $xy$-plane, the planes containing the $z$-axis and the circular cylinders with axis the $z$-axis.



5.5.2 By considering the quadric surface $F_t(x, y, z) = 0$, where

$$F_t(x, y, z) = \frac{x^2}{p^2 - t} + \frac{y^2}{q^2 - t} - 2z + t,$$

construct a triply orthogonal system (illustrated above) consisting of two families of elliptic paraboloids and one family of hyperbolic paraboloids. Find a parametrization of these surfaces analogous to (5.12).

5.5.3 Show that the following are triply orthogonal systems:
(i) $xy = uz^2$, $x^2 + y^2 + z^2 = v$, $x^2 + y^2 + z^2 = w(x^2 - y^2)$.
(ii) $yz = ux$, $\sqrt{x^2 + y^2} + \sqrt{x^2 + z^2} = v$, $\sqrt{x^2 + y^2} - \sqrt{x^2 + z^2} = w$.

5.5.4 What should be the definition of a (doubly) orthogonal system of curves in $\mathbb{R}^2$? Give examples of such systems such that:

(i) Each of the two families of curves consists of parallel straight lines.
(ii) One family consists of straight lines and the other consists of circles.

5.5.5 By considering the function

$$F_t(x,y) = \frac{x^2}{p^2-t} + \frac{y^2}{q^2-t},$$

where $p$ and $q$ are constants with $0 < p^2 < q^2$, construct an orthogonal system of curves in which one family consists of ellipses and the other consists of hyperbolas.
By imitating Exercise 5.5.2, construct in a similar way an orthogonal system of curves in which both families consist of parabolas.

5.5.6 Starting with an orthogonal system of curves in the $xy$-plane, construct two families of generalized cylinders with axis parallel to the $z$-axis which intersect the $xy$-plane in the two given families of curves. Show that these two families of cylinders, together with the planes parallel to the $xy$-plane, form a triply-orthogonal system.

5.6.1 Show that, if $\boldsymbol{\gamma} : (\alpha, \beta) \to \mathbb{R}^3$ is a curve whose image is contained in a surface patch $\boldsymbol{\sigma} : U \to \mathbb{R}^3$, then $\boldsymbol{\gamma}(t) = \boldsymbol{\sigma}(u(t), v(t))$ for some smooth map $(\alpha, \beta) \to U$, $t \mapsto (u(t), v(t))$.

5.6.2 Prove Theorem 1.5.1 and its analogue for level curves in $\mathbb{R}^3$ (Exercise 1.5.1).

5.6.3 Let $\boldsymbol{\sigma} : U \to \mathbb{R}^3$ be a smooth map such that $\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v \neq \mathbf{0}$ at some point $(u_0, v_0) \in U$. Show that there is an open subset $W$ of $U$ containing $(u_0, v_0)$ such that the restriction of $\boldsymbol{\sigma}$ to $W$ is injective. Note that, in the text, surface patches are injective by definition, but this exercise shows that injectivity near a given point is a consequence of regularity.

5.6.4 Let $\boldsymbol{\sigma} : U \to \mathbb{R}^3$ be a regular surface patch, let $(u_0, v_0) \in U$ and let $\boldsymbol{\sigma}(u_0, v_0) = (x_0, y_0, z_0)$. Suppose that the unit normal $\mathbf{N}(u_0, v_0)$ is not parallel to the $xy$-plane. Show that there is an open set $V$ in $\mathbb{R}^2$ containing $(x_0, y_0)$, an open subset $W$ of $U$ containing $(u_0, v_0)$ and a smooth function $\varphi : V \to \mathbb{R}$ such that $\tilde{\boldsymbol{\sigma}}(x, y) = (x, y, \varphi(x, y))$ is a reparametrization of $\boldsymbol{\sigma} : W \to \mathbb{R}^3$. Thus, 'near' $\mathbf{p}$, the surface is part of the graph $z = \varphi(x, y)$.
What happens if $\mathbf{N}(u_0, v_0)$ is parallel to the $xy$-plane?

5.6.5 Let $\boldsymbol{\gamma} : (\alpha, \beta) \to \mathbb{R}^n$ be a regular curve and let $t_0 \in (\alpha, \beta)$. Show that, for some $\epsilon > 0$, the restriction of $\boldsymbol{\gamma}$ to the subinterval $(t_0 - \epsilon, t_0 + \epsilon)$ of $(\alpha, \beta)$ is injective.

# Chapter 6

6.1.1 Calculate the first fundamental forms of the following surfaces:

(i) $\boldsymbol{\sigma}(u,v) = (\sinh u \sinh v, \sinh u \cosh v, \sinh u)$.
(ii) $\boldsymbol{\sigma}(u,v) = (u-v, u+v, u^2+v^2)$.
(iii) $\boldsymbol{\sigma}(u,v) = (\cosh u, \sinh u, v)$.
(iv) $\boldsymbol{\sigma}(u,v) = (u, v, u^2+v^2)$.
What kinds of surfaces are these ?

6.1.2 Show that applying an isometry of $\mathbb{R}^3$ to a surface does not change its first fundamental form. What is the effect of a dilation (i.e. a map $\mathbb{R}^3 \to \mathbb{R}^3$ of the form $\mathbf{v} \mapsto a\mathbf{v}$ for some constant $a \neq 0$)?

6.1.3 Let $Edu^2+2Fdudv+Gdv^2$ be the first fundamental form of a surface patch $\boldsymbol{\sigma}(u,v)$ of a surface $\mathcal{S}$. Show that, if $\mathbf{p}$ is a point in the image of $\boldsymbol{\sigma}$ and $\mathbf{v},\mathbf{w} \in T_{\mathbf{p}}\mathcal{S}$, then

$$\langle \mathbf{v},\mathbf{w}\rangle = Edu(\mathbf{v})du(\mathbf{w}) + F(du(\mathbf{v})dv(\mathbf{w}) + du(\mathbf{w})dv(\mathbf{v})) + Gdu(\mathbf{w})dv(\mathbf{w}).$$

6.1.4 Suppose that a surface patch $\tilde{\boldsymbol{\sigma}}(\tilde{u},\tilde{v})$ is a reparametrization of a surface patch $\boldsymbol{\sigma}(u,v)$, and let

$$\tilde{E}d\tilde{u}^2 + 2\tilde{F}d\tilde{u}d\tilde{v} + \tilde{G}d\tilde{v}^2 \quad \text{and} \quad Edu^2 + 2Fdudv + Gdv^2$$

be their first fundamental forms. Show that:
(i) $du = \frac{\partial u}{\partial \tilde{u}}d\tilde{u} + \frac{\partial u}{\partial \tilde{v}}d\tilde{v}, \quad dv = \frac{\partial v}{\partial \tilde{u}}d\tilde{u} + \frac{\partial v}{\partial \tilde{v}}d\tilde{v}$.
(ii) If

$$J = \begin{pmatrix} \frac{\partial u}{\partial \tilde{u}} & \frac{\partial u}{\partial \tilde{v}} \\ \frac{\partial v}{\partial \tilde{u}} & \frac{\partial v}{\partial \tilde{v}} \end{pmatrix}$$

is the Jacobian matrix of the reparametrization map $(\tilde{u},\tilde{v}) \mapsto (u,v)$, and $J^t$ is the transpose of $J$, then

$$\begin{pmatrix} \tilde{E} & \tilde{F} \\ \tilde{F} & \tilde{G} \end{pmatrix} = J^t \begin{pmatrix} E & F \\ F & G \end{pmatrix} J.$$

6.1.5 Show that the following are equivalent conditions on a surface patch $\boldsymbol{\sigma}(u,v)$ with first fundamental form $Edu^2+2Fdudv+Gdv^2$:
(i) $E_v = G_u = 0$.
(ii) $\boldsymbol{\sigma}_{uv}$ is parallel to the standard unit normal $\mathbf{N}$.
(iii) The opposite sides of any quadrilateral formed by parameter curves of $\boldsymbol{\sigma}$ have the same length (see the remarks following the proof of Proposition 4.4.2).
When these conditions are satisfied, the parameter curves of $\boldsymbol{\sigma}$ are said to form a *Chebyshev net*. Show that, in that case, $\boldsymbol{\sigma}$ has a reparametrization $\tilde{\boldsymbol{\sigma}}(\tilde{u},\tilde{v})$ with first fundamental form

$$d\tilde{u}^2 + 2\cos\theta\, d\tilde{u}d\tilde{v} + d\tilde{v}^2,$$

where $\theta$ is a smooth function of $(\tilde{u}, \tilde{v})$. Show that $\theta$ is the angle between the parameter curves of $\tilde{\boldsymbol{\sigma}}$. Show further that, if we put $\hat{u} = \tilde{u} + \tilde{v}$, $\hat{v} = \tilde{u} - \tilde{v}$, the resulting reparametrization $\hat{\boldsymbol{\sigma}}(\hat{u}, \hat{v})$ of $\tilde{\boldsymbol{\sigma}}(\tilde{u}, \tilde{v})$ has first fundamental form

$$\cos^2 \omega \, d\hat{u}^2 + \sin^2 \omega \, d\hat{v}^2,$$

where $\omega = \theta/2$.

6.1.6 Repeat Exercise 6.1.1 for the following surfaces:
(i) $\boldsymbol{\sigma}(u, v) = (u \cos v, u \sin v, \ln u)$.
(ii) $\boldsymbol{\sigma}(u, v) = (u \cos v, u \sin v, v)$.
(iii) $\boldsymbol{\sigma}(u, v) = (\cosh u \cos v, \cosh u \sin v, u)$.

6.1.7 Find the length of the part of the curve on the cone in Exercise 2.3.13 with $0 \leq t \leq \pi$. Show also that the curve intersects each of the rulings of the cone at the same angle.

6.1.8 Let $\boldsymbol{\sigma}$ be the ruled surface generated by the binormals $\mathbf{b}$ of a unit-speed curve $\boldsymbol{\gamma}$:

$$\boldsymbol{\sigma}(u, v) = \boldsymbol{\gamma}(u) + v\mathbf{b}(u).$$

Show that the first fundamental form of $\boldsymbol{\sigma}$ is

$$(1 + v^2\tau^2)\, du^2 + dv^2,$$

where $\tau$ is the torsion of $\boldsymbol{\gamma}$.

6.1.9 If $E$, $F$ and $G$ are the coefficients of the first fundamental form of a surface patch $\boldsymbol{\sigma}(u, v)$, show that $E_u = 2\boldsymbol{\sigma}_u . \boldsymbol{\sigma}_{uu}$, and find similar expressions for $E_v$, $F_u$, $F_v$, $G_u$ and $G_v$. Deduce the following formulas:

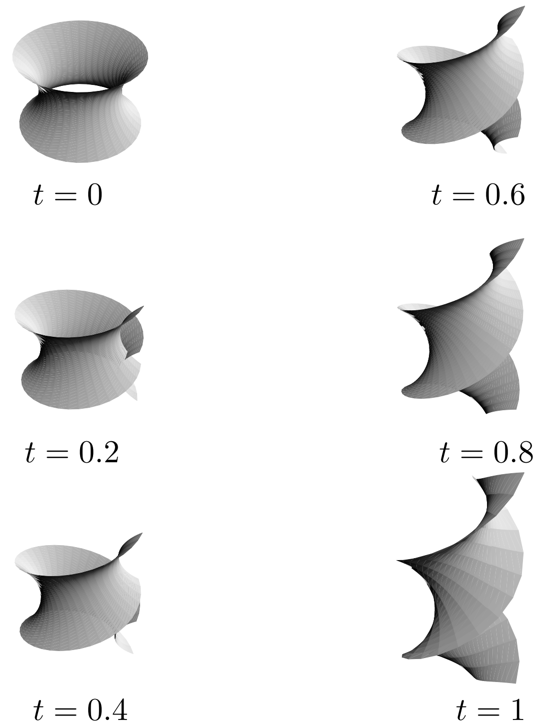
$$\begin{aligned}
\boldsymbol{\sigma}_u . \boldsymbol{\sigma}_{uu} &= \frac{1}{2}E_u, \quad & \boldsymbol{\sigma}_v . \boldsymbol{\sigma}_{uu} &= F_u - \frac{1}{2}E_v \\
\boldsymbol{\sigma}_u . \boldsymbol{\sigma}_{uv} &= \frac{1}{2}E_v, \quad & \boldsymbol{\sigma}_v . \boldsymbol{\sigma}_{uu} &= \frac{1}{2}G_u \\
\boldsymbol{\sigma}_u . \boldsymbol{\sigma}_{vv} &= F_v - \frac{1}{2}G_u, \quad & \boldsymbol{\sigma}_v . \boldsymbol{\sigma}_{uu} &= \frac{1}{2}G_v.
\end{aligned}$$

6.2.1 By thinking about how a circular cone can be 'unwrapped' onto the plane, write down an isometry from

$$\boldsymbol{\sigma}(u, v) = (u \cos v, u \sin v, u), \qquad u > 0, \ 0 < v < 2\pi,$$

(a circular cone with a straight line removed) to an open subset of the $xy$-plane.

6.2.2 Is the map from the circular half-cone $x^2 + y^2 = z^2$, $z > 0$, to the $xy$-plane given by $(x, y, z) \mapsto (x, y, 0)$ a local isometry?




6.2.3 Consider the surface patches

$$\boldsymbol{\sigma}(u,v) = (\cosh u \cos v, \cosh u \sin v, u), \quad \tilde{\boldsymbol{\sigma}}(u,v) = (u\cos v, u \sin v, v),$$

parametrizing the catenoid (Exercise 5.3.1) and the helicoid (Exercise 4.2.6), respectively. Show that the map from the catenoid to the helicoid that takes $\boldsymbol{\sigma}(u,v)$ to $\tilde{\boldsymbol{\sigma}}(\sinh u, v)$ is a local isometry. Which curves on the helicoid correspond under this isometry to the parallels and meridians of the catenoid ?

In fact, there is an *isometric deformation* of the catenoid into a helicoid. Let

$$\hat{\boldsymbol{\sigma}}(u,v) = (-\sinh u \sin v, \sinh u \cos v, -v).$$

This is the result of reflecting the helicoid $\tilde{\boldsymbol{\sigma}}$ in the $xy$-plane and then translating it by $\pi/2$ parallel to the $z$-axis. Define

$$\boldsymbol{\sigma}^t(u,v) = \cos t\, \boldsymbol{\sigma}(u,v) + \sin t\, \hat{\boldsymbol{\sigma}}(u,v),$$

so that $\boldsymbol{\sigma}^0(u,v) = \boldsymbol{\sigma}(u,v)$ and $\boldsymbol{\sigma}^{\pi/2}(u,v) = \hat{\boldsymbol{\sigma}}(u,v)$. Show that, for all values of $t$, the map $\boldsymbol{\sigma}(u,v) \mapsto \boldsymbol{\sigma}^t(u,v)$ is a local isometry. Show also that the tangent plane of $\boldsymbol{\sigma}^t$ at the point $\boldsymbol{\sigma}^t(u,v)$ depends only of $u, v$ and not on $t$. The surfaces $\boldsymbol{\sigma}^t$ are shown above for several values of $t$. (The result of this exercise is 'explained' in Exercises 12.5.3 and 12.5.4.)

6.2.4 Show that the line of striction (Exercise 5.3.4) of the tangent developable of a unit-speed curve $\boldsymbol{\gamma}$ is $\boldsymbol{\gamma}$ itself. Show also that the intersection of this surface with the plane passing through a point $\boldsymbol{\gamma}(u_0)$ of the curve and perpendicular to it at that point is a curve of the form

$$\boldsymbol{\Gamma}(v) = \boldsymbol{\gamma}(u_0) - \frac{1}{2}\kappa(u_0)v^2\mathbf{n}(u_0) + \frac{1}{3}\kappa(u_0)\tau(u_0)v^3\mathbf{b}(u_0)$$

if we neglect higher powers of $v$ (we assume that the curvature $\kappa(u_0)$ and the torsion $\tau(u_0)$ of $\boldsymbol{\gamma}$ at $\boldsymbol{\gamma}(u_0)$ are both non-zero). Note that this curve has an ordinary cusp (Exercise 1.3.3) at $\boldsymbol{\gamma}(u_0)$, so the tangent developable has a sharp 'edge' where the two sheets $v > 0$ and $v < 0$ meet along $\boldsymbol{\gamma}$. This is evident for the tangent developable of a circular helix illustrated earlier in this section.

6.2.5 Show that every generalized cylinder and every generalized cone is locally isometric to a plane.

6.2.6 Suppose that a surface patch $\boldsymbol{\sigma}$ has first fundamental form

$$du^2 + f(u)^2 dv^2,$$

where $f$ is a smooth function of $u$ only. Show that, if

$$\left|\frac{df}{du}\right| < 1 \text{ for all values of } u,$$

then $\boldsymbol{\sigma}$ is locally isometric to a surface of revolution.

6.2.7 Suppose that a surface $\boldsymbol{\sigma}$ has first fundamental form

$$U(du^2 + dv^2),$$

where $U$ is a smooth function of $u$ only. Show that $\boldsymbol{\sigma}$ is isometric to a surface of revolution if

$$\left|\frac{dU}{du}\right| < 2U$$

for all values of $u$.

6.3.1 Show that every local isometry is conformal. Give an example of a conformal map that is not a local isometry.

6.3.2 Show that *Enneper's surface*

$$\boldsymbol{\sigma}(u, v) = \left(u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + vu^2, u^2 - v^2\right)$$

is conformally parametrized.

6.3.3 Recall from Example 6.1.3 that the first fundamental form of the latitude-longitude parametrization $\boldsymbol{\sigma}(\theta, \varphi)$ of $S^2$ is

$$d\theta^2 + \cos^2\theta\, d\varphi^2.$$

Find a smooth function $\psi$ such that the reparametrization $\tilde{\boldsymbol{\sigma}}(u, v) = \boldsymbol{\sigma}(\psi(u), v)$ is conformal. Verify that $\tilde{\boldsymbol{\sigma}}$ is, in fact, the Mercator parametrization in Exercise 5.3.2.

6.3.4 Let $\Phi : U \to V$ be a diffeomorphism between open subsets of $\mathbb{R}^2$. Write

$$\Phi(u, v) = (f(u, v), g(u, v)),$$

where $f$ and $g$ are smooth functions on the $uv$-plane. Show that $\Phi$ is conformal if and only if

$$\text{(6.11)} \qquad \text{either } \ (f_u = g_v \ \text{ and } \ f_v = -g_u) \ \text{ or } \ (f_u = -g_v \ \text{ and } \ f_v = g_u).$$

Show that, if $J(\Phi)$ is the Jacobian matrix of $\Phi$, then $\det(J(\Phi)) > 0$ in the first case and $\det(J(\Phi)) < 0$ in the second case.

6.3.5 (This exercise requires a basic knowledge of complex analysis.) Recall that the transition map between two surface patches in an atlas for a surface $\mathcal{S}$ is a smooth map between open subsets of $\mathbb{R}^2$. Since $\mathbb{R}^2$ is the 'same' as the complex numbers $\mathbb{C}$ (via $(u, v) \leftrightarrow u + iv$), we can ask whether such a transition map is *holomorphic*. One says that $\mathcal{S}$ is a *Riemann surface* if $\mathcal{S}$ has an atlas for which all the transition maps are holomorphic. Deduce from Theorem 6.3.6 and the preceding exercise that every orientable surface has an atlas making it a Riemann surface. (You will need to recall from complex analysis that a smooth function $\Phi$ as in the preceding exercise is holomorphic if and only if the first pair of equations in (6.11) hold - these are the *Cauchy-Riemann equations*. If the second pair of equations in (6.11) hold, $\Phi$ is said to be *anti-holomorphic*.)

6.3.6 Define a map $\tilde{\Pi}$ similar to $\Pi$ by projecting from the *south* pole of $S^2$ onto the $xy$-plane. Show that this defines a second conformal surface patch $\tilde{\boldsymbol{\sigma}}_1$ which covers the whole of $S^2$ except the south pole. What is the transition map between these two patches? Why do the two patches $\boldsymbol{\sigma}_1$ and $\tilde{\boldsymbol{\sigma}}_1$ *not* give $S^2$ the structure of a Riemann surface? How can $\tilde{\boldsymbol{\sigma}}_1$ be modified to produce such a structure?

6.3.7 Show that the stereographic projection map $\Pi$ takes circles on $S^2$ to Circles in $\mathbb{C}_\infty$, and that every Circle arises in this way. (A circle on $S^2$ is the intersection of $S^2$ with a plane; a Circle in $\mathbb{C}_\infty$ is a line or a circle in $\mathbb{C}$ - see Appendix 2.)

6.3.8 Show that, if $M$ is a Möbius transformation or a conjugate-Möbius transformation (see Appendix 2), the bijection $\Pi^{-1} \circ M \circ \Pi : S^2 \to S^2$ is a conformal diffeomorphism of $S^2$. It can be shown that every conformal diffeomorphism of $S^2$ is of this type.

6.3.9 Let $f$ be a smooth function and let

$$\boldsymbol{\sigma}(u, v) = (u \cos v, u \sin v, f(u))$$

be the surface obtained by rotating the curve $z = f(x)$ in the $xz$-plane around the $z$-axis. Find all functions $f$ for which $\boldsymbol{\sigma}$ is conformal.

6.3.10 Let $\boldsymbol{\sigma}$ be the ruled surface

$$\boldsymbol{\sigma}(u, v) = \boldsymbol{\gamma}(u) + v\boldsymbol{\delta}(u),$$

where $\boldsymbol{\gamma}$ is a unit-speed curve in $\mathbb{R}^3$ and $\boldsymbol{\delta}(u)$ is a unit vector for all $u$. Prove that $\boldsymbol{\sigma}$ is conformal if and only if $\boldsymbol{\delta}$ is constant and $\boldsymbol{\gamma}$ lies in a plane perpendicular to $\boldsymbol{\delta}$. What kind of surface is $\boldsymbol{\sigma}$ in this case?

6.3.11 With the notation in Exercise 4.5.5, show that the inversion map $F : \mathcal{S} \to \mathcal{S}^*$ is conformal.

6.3.12 Show (without using Theorem 6.3.6!) that every surface of revolution has an atlas consisting of conformal surface patches.

6.4.1 Determine the area of the part of the paraboloid $z = x^2 + y^2$ with $z \leq 1$ and compare with the area of the hemisphere $x^2 + y^2 + z^2 = 1$, $z \leq 0$.

6.4.2 A sailor circumnavigates Australia by a route consisting of a triangle whose sides are arcs of great circles. Prove that at least one interior angle of the triangle is $\geq \frac{\pi}{3} + \frac{10}{169}$ radians. (Take the Earth to be a sphere of radius 6500km and assume that the area of Australia is 7.5 million square km.)

6.4.3 A *spherical polygon* on $S^2$ is the region formed by the intersection of $n$ hemispheres of $S^2$, where $n$ is an integer $\geq 3$. Show that, if $\alpha_1, \ldots, \alpha_n$ are the interior angles of such a polygon, its area is equal to

$$\sum_{i=1}^{n} \alpha_i - (n-2)\pi.$$

6.4.4 Suppose that $S^2$ is covered by spherical polygons, and such that the intersection of any two polygons is either empty or a common edge or vertex of each polygon. Suppose that there are $F$ polygons, $E$ edges and $V$ vertices (a common edge or vertex of more than one polygon being counted only once). Show that the sum of the angles of all the polygons is $2\pi V$. By using the preceding exercise, deduce that $V - E + F = 2$. (This result is due to Euler; it is generalized in Chapter 13.)

6.4.5 Show that:
(i) Every local isometry is an equiareal map.
(ii) A map that is both conformal and equiareal is a local isometry.

Give an example of an equiareal map that is not a local isometry.

6.4.6 Prove Theorem 6.4.5.

6.4.7 Let $\boldsymbol{\sigma}(u,v)$ be a surface patch with standard unit normal $\mathbf{N}$. Show that

$$\mathbf{N}\times\boldsymbol{\sigma}_u = \frac{E\boldsymbol{\sigma}_v - F\boldsymbol{\sigma}_u}{\sqrt{EG-F^2}}, \quad \mathbf{N}\times\boldsymbol{\sigma}_v = \frac{F\boldsymbol{\sigma}_v - G\boldsymbol{\sigma}_v}{\sqrt{EG-F^2}}.$$

6.4.8 Find the area of the part of the helicoid

$$\boldsymbol{\sigma}(u,v) = (u\cos v, u\sin v, v)$$

corresponding to $0 < u < 1$, $0 < v < 2\pi$.

6.4.9 Suppose that all the polygons in Exercise 6.4.4 have the same number $n$ of edges, and that the same number $m$ of polygons meet at each vertex. Show that $nF = 2E = mV$ and hence find $V$, $E$ and $F$ in terms of $m$ and $n$. Show that $1/m + 1/n > 1/2$ and deduce that there are exactly five possibilities for the pair $(m,n)$.
A *polyhedron* is a convex subset of $\mathbb{R}^3$ bounded by a finite number of plane polygons. Take a point $\mathbf{p}$ inside such a polyhedron and for any point $\mathbf{q}$ on an edge of the polyhedron draw the straight line through $\mathbf{p}$ and $\mathbf{q}$. This line intersects the sphere with centre $\mathbf{p}$ and radius 1 in a point $\mathbf{v}$, say. The collection of such points $\mathbf{v}$ form the edges of a covering of the sphere with spherical polygons as in the first part. The result of this exercise therefore gives a classification of polyhedra such that all faces have the same number of sides and the same number of edges meet at each vertex. (Note that it is not necessary to assume that the polyhedron is *regular*, i.e. that all the edges have the same length.)

6.4.10 Show that, given 5 points on a sphere, it is impossible to connect each pair by curves on the sphere that intersect only at the given points. Deduce that the same result holds if 'sphere' is replaced by 'plane'.

6.4.11 Let $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ and $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$ be points on a sphere. Show that it is impossible to join each $\mathbf{p}_i$ to each $\mathbf{q}_j$ by nine curves on the sphere that intersect only at the given points. (This is sometimes called the 'Utilities Problem', thinking of $\mathbf{p}_1$, $\mathbf{p}_2$ and $\mathbf{p}_3$ as the gas, water and electricity supplies to three homes $\mathbf{q}_1$, $\mathbf{q}_2$ and $\mathbf{q}_3$.)

6.4.12 A surface is obtained by rotating about the $z$-axis a unit-speed curve $\boldsymbol{\gamma}$ in the $xz$-plane that does not intersect the $z$-axis. Show that its area is

$$2\pi \int \rho(u)\, du,$$

where $\rho(u)$ is the distance of $\boldsymbol{\gamma}(u)$ from the $z$-axis. Hence find the area of

(i) $S^2$;
(ii) the torus in Exercise 4.2.5.

6.4.13 Prove that the area of the part of the tube of radius $a$ around a curve $\boldsymbol{\gamma}(s)$ given by $s_0 < s < s_1$, where $s_0$ and $s_1$ are constants, is $2\pi a(s_1 - s_0)$. (See Exercise 4.2.7.)

6.4.14 Show that a map between surfaces that is both conformal and equiareal is a local isometry.

6.4.15 Let $M : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map, and let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$ be the images under $M$ of the vectors $\mathbf{i} = (1, 0)$, $\mathbf{j} = (0, 1)$. Show that:
(i) $M$ is a diffeomorphism if and only if $\mathbf{u}$ and $\mathbf{v}$ are linearly independent.
(ii) $M$ is an isometry if and only if $\mathbf{u}$ and $\mathbf{v}$ are perpendicular unit vectors.
(iii) $M$ is conformal if and only if $\mathbf{u}$ and $\mathbf{v}$ are perpendicular vectors of equal length.
(iv) $M$ is equiareal if and only if $\mathbf{u} \times \mathbf{v}$ is a unit vector.

6.4.16 Find all functions $f$ for which the surface patch $\boldsymbol{\sigma}$ in Exercise 6.3.9 is equiareal.

6.4.17 In the notation of the proof of Theorem 6.4.6, let

$$\boldsymbol{\sigma}_3(\theta, \varphi) = (\cos\varphi, \sin\varphi, \sin\theta + f(\varphi)),$$

where $f$ is any $2\pi$-periodic smooth function. Show that the map $\boldsymbol{\sigma}_1(\theta, \varphi) \mapsto \boldsymbol{\sigma}_3(\theta, \varphi)$ from $S^2$ to the unit cylinder is equiareal.

6.5.1 Find the angles and the lengths of the sides of an equilateral spherical triangle whose area is one quarter of the area of the sphere.

6.5.2 Show that similar spherical triangles are congruent.

6.5.3 The *spherical circle* of centre $\mathbf{p} \in S^2$ and radius $R$ is the set of points of $S^2$ that are a spherical distance $R$ from $\mathbf{p}$. Show that, if $0 \le R \le \pi/2$:
(i) A spherical circle of radius $R$ is a circle of radius $\sin R$.
(ii) The area inside a spherical circle of radius $R$ is $2\pi(1 - \cos R)$.
What if $R > \pi/2$?

6.5.4 This exercise describes the transformations of $\mathbb{C}_\infty$ corresponding to the isometries of $S^2$ under the stereographic projection map $\Pi : S^2 \to \mathbb{C}_\infty$ (Example 6.3.5). If $F$ is any isometry of $S^2$, let $F_\infty = \Pi \circ F \circ \Pi^{-1}$ be the corresponding bijection $C_\infty \to \mathbb{C}_\infty$.
(i) A Möbius transformation

$$M(w) = \frac{aw + b}{cw + d},$$

where $a, b, c, d \in \mathbb{C}$ and $ad - bc \ne 0$, is said to be *unitary* if $d = \bar{a}$ and $c = -\bar{b}$ (see Appendix 2). Show that the composite of two unitary Möbius transformations is unitary and that the inverse of a unitary Möbius transformation is unitary.

(ii) Show that if $F$ is the reflection in the plane passing through the origin and perpendicular to the unit vector $(a, b)$ (where $a \in \mathbb{C}$, $b \in \mathbb{R}$ - see Example 5.3.4), then

$$F_\infty(w) = \frac{-a\bar{w} + b}{b\bar{w} + \bar{a}}.$$

(iii) Deduce that if $F$ is any isometry of $S^2$ there is a unitary Möbius transformation $M$ such that either $F_\infty = M$ or $F_\infty = M \circ C$ where $J(w) = -\bar{w}$.
(iv) Show conversely that if $M$ is any unitary Möbius transformation, the bijections $\mathbb{C}_\infty \to \mathbb{C}_\infty$ given by $M$ and $M \circ J$ are both of the form $F_\infty$ for some isometry $F$ of $S^2$.

6.5.5 What does the cosine rule become for a triangle on a sphere of radius $R$? Explain how and why this becomes the cosine rule for the plane when $R \to \infty$.

6.5.6 Find the distance between Athens (latitude 38°, longitude 24°) and Bombay (latitude 19°, longitude 73°) measured along the short great circle arc joining them. (Take the radius of the Earth to be 6500km.)

6.5.7 A spherical square on $S^2$ has each side of length $A$ and each angle equal to $\alpha$ (each side being an arc of a great circle). Show that

$$\cos A = \cot^2 \frac{1}{2}\alpha.$$

6.5.8 In the notation of Proposition 6.5.3, let $\lambda = \sin\alpha / \sin A$.
(i) Show that $\sin\alpha + \sin\beta = \lambda(\sin A + \sin B)$, $\sin\alpha - \sin\beta = \lambda(\sin A - \sin B)$.
(ii) Show that $\cos\alpha + \cos\beta\cos\gamma = \lambda\sin\gamma\sin B\cos A$, and obtain a similar formula for $\cos\beta + \cos\alpha\cos\gamma$.
(iii) Deduce from (ii) that

$$(\cos\alpha + \cos\beta)(1 + \cos\gamma) = \lambda\sin\gamma\sin(A + B).$$

(iv) Deduce from (i) and (iii) that

$$\tan\frac{1}{2}(\alpha + \beta) = \frac{\cos\frac{1}{2}(A - B)}{\cos\frac{1}{2}(A + B)}\cot\frac{1}{2}\gamma$$

and prove similarly that

$$\tan\frac{1}{2}(\alpha - \beta) = \frac{\sin\frac{1}{2}(A - B)}{\sin\frac{1}{2}(A + B)}\cot\frac{1}{2}\gamma.$$

(v) Find two formulas similar to those in (iv) for $\tan\frac{1}{2}(A \pm B)$.
The formulas in (iv) and (v) are called *Napier's Analogies* (after the same Napier who invented logarithms).

6.5.9 Suppose that two spherical triangles with vertices $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and $\mathbf{a}', \mathbf{b}', \mathbf{c}'$ are such that
(i) the angle of the triangles at $\mathbf{a}$ and $\mathbf{a}'$ are equal, and
(ii) the two sides of the first triangle meeting at $\mathbf{a}$ have the same length as the two sides of the second triangle meeting at $\mathbf{a}'$.
Prove that the triangles are congruent.

# Chapter 7

7.1.1 Compute the second fundamental form of the elliptic paraboloid

$$\boldsymbol{\sigma}(u, v) = (u, v, u^2 + v^2).$$

7.1.2 Suppose that the second fundamental form of a surface patch $\boldsymbol{\sigma}$ is zero everywhere. Prove that $\boldsymbol{\sigma}$ is an open subset of a plane. This is the analogue for surfaces of the theorem that a curve with zero curvature everywhere is part of a straight line.

7.1.3 Let a surface patch $\tilde{\boldsymbol{\sigma}}(\tilde{u}, \tilde{v})$ be a reparametrization of a surface patch $\boldsymbol{\sigma}(u, v)$ with reparametrization map $(u, v) = \Phi(\tilde{u}, \tilde{v})$. Prove that

$$\begin{pmatrix} \tilde{L} & \tilde{M} \\ \tilde{M} & \tilde{N} \end{pmatrix} = \pm J^t \begin{pmatrix} L & M \\ M & N \end{pmatrix} J,$$

where $J$ is the Jacobian matrix of $\Phi$ and we take the plus sign if $\det(J) > 0$ and the minus sign if $\det(J) < 0$. Deduce from Exercise 6.1.4 that the second fundamental form of a surface patch is unchanged by a reparametrization of the patch which preserves its orientation.

7.1.4 What is the effect on the second fundamental form of a surface of applying an isometry of $\mathbb{R}^3$? Or a dilation?

7.1.5 Repeat Exercise 7.1.1 for the helicoid

$$\boldsymbol{\sigma}(u, v) = (u \cos v, u \sin v, v).$$

7.1.6 Find the second fundamental form of the tangent developable of a unit-speed curve $\boldsymbol{\gamma}$ with nowhere vanishing curvature (see §6.2). Show that the second fundamental form is zero everywhere if and only if $\boldsymbol{\gamma}$ is planar. How is this result related to Exercise 7.1.2?

7.2.1 Calculate the Gauss map $\mathcal{G}$ of the paraboloid $\mathcal{S}$ with equation $z = x^2 + y^2$. What is the image of $\mathcal{G}$?

7.2.2 Show that the Weingarten map changes sign when the orientation of the surface changes.

7.2.3 Repeat Exercise 7.2.1 for the hyperboloid of one sheet $x^2 + y^2 - z^2 = 1$ and the hyperboloid of two sheets $x^2 - y^2 - z^2 = 1$.

7.3.1 Let $\boldsymbol{\gamma}$ be a regular, but not necessarily unit-speed, curve on a surface. Prove that (with the usual notation) the normal and geodesic curvatures of $\boldsymbol{\gamma}$ are

$$\kappa_n = \frac{\langle\langle \dot{\boldsymbol{\gamma}}, \dot{\boldsymbol{\gamma}} \rangle\rangle}{\langle \dot{\boldsymbol{\gamma}}, \dot{\boldsymbol{\gamma}} \rangle} \quad \text{and} \quad \kappa_g = \frac{\ddot{\boldsymbol{\gamma}}.(\mathbf{N} \times \dot{\boldsymbol{\gamma}})}{\langle \dot{\boldsymbol{\gamma}}, \dot{\boldsymbol{\gamma}} \rangle^{3/2}}.$$

7.3.2 Show that the normal curvature of any curve on a sphere of radius $r$ is $\pm 1/r$.

7.3.3 Compute the geodesic curvature of any circle on a sphere (not necessarily a great circle).

7.3.4 Show that, if $\boldsymbol{\gamma}(t) = \boldsymbol{\sigma}(u(t), v(t))$ is a unit-speed curve on a surface patch $\boldsymbol{\sigma}$ with first fundamental form $Edu^2 + 2Fdudv + Gdv^2$, the geodesic curvature of $\boldsymbol{\gamma}$ is

$$\kappa_g = (\ddot{v}\dot{u} - \dot{v}\ddot{u})\sqrt{EG - F^2} + A\dot{u}^3 + B\dot{u}^2\dot{v} + C\dot{u}\dot{v}^2 + D\dot{v}^3,$$

where $A$, $B$, $C$ and $D$ can be expressed in terms of $E$, $F$, $G$ and their derivatives. Find $A, B, C, D$ explicitly when $F = 0$.

7.3.5 Suppose that a unit-speed curve $\boldsymbol{\gamma}$ with curvature $\kappa > 0$ and principal normal $\mathbf{n}$ is a parametrization of the intersection of two oriented surfaces $\mathcal{S}_1$ and $\mathcal{S}_2$ with unit normals $\mathbf{N}_1$ and $\mathbf{N}_2$. Show that, if $\kappa_1$ and $\kappa_2$ are the normal curvatures of $\boldsymbol{\gamma}$ when viewed as a curve in $\mathcal{S}_1$ and $\mathcal{S}_2$, respectively, then

$$\kappa_1 \mathbf{N}_2 - \kappa_2 \mathbf{N}_1 = \kappa(\mathbf{N}_1 \times \mathbf{N}_2) \times \mathbf{n}.$$

Deduce that, if $\alpha$ is the angle between the two surfaces,

$$\kappa^2 \sin^2 \alpha = \kappa_1^2 + \kappa_2^2 - 2\kappa_1\kappa_2 \cos \alpha.$$

7.3.6 A curve $\boldsymbol{\gamma}$ on a surface $\mathcal{S}$ is called *asymptotic* if its normal curvature is everywhere zero. Show that any straight line on a surface is an asymptotic curve. Show also that a curve $\boldsymbol{\gamma}$ with positive curvature is asymptotic if and only if its binormal $\mathbf{b}$ is parallel to the unit normal of $\mathcal{S}$ at all points of $\boldsymbol{\gamma}$.

7.3.7 Prove that the asymptotic curves on the surface

$$\boldsymbol{\sigma}(u, v) = (u \cos v, u \sin v, \ln u)$$

are given by

$$\ln u = \pm(v + c),$$

where $c$ is an arbitrary constant.

7.3.8 Show that if a curve on a surface has zero normal and geodesic curvature everywhere, it is part of a straight line.

7.3.9 Calculate the normal curvature at the point $(1,0,1)$ of the curve $\boldsymbol{\gamma}$ on the hyperbolic paraboloid

$$\boldsymbol{\sigma}(u,v) = \left(\frac{1}{2}(u+v), \frac{1}{2}(v-u), uv\right)$$

corresponding to the straight line $u = v = t$ in the $uv$-plane (note that $\boldsymbol{\gamma}$ is *not* unit-speed).

7.3.10 Consider the surface of revolution

$$\boldsymbol{\sigma}(u,v) = (f(u)\cos v, f(u)\sin v, g(u)),$$

where $u \mapsto (f(u), 0, g(u))$ is a unit-speed curve in $\mathbb{R}^3$. Compute the geodesic and normal curvatures of

(i) a meridian $v =$ constant;
(ii) a parallel $u =$ constant.

7.3.11 Find the geodesic and normal curvatures of a circle $z =$ constant on the paraboloid $x^2 + y^2 = z$.

7.3.12 Consider the ruled surface

$$\boldsymbol{\sigma}(u,v) = \boldsymbol{\gamma}(u) + v\boldsymbol{\delta}(u),$$

where $\boldsymbol{\gamma}$ is a unit-speed curve and $\boldsymbol{\delta}$ is a unit vector. Show that the geodesic curvature of the curve $\boldsymbol{\gamma}$ on $\boldsymbol{\sigma}$ is

$$\kappa_g = -\dot{\theta} - \frac{\mathbf{t}.\dot{\boldsymbol{\delta}}}{\sin\theta},$$

where $\mathbf{t}$ is the tangent vector of $\boldsymbol{\gamma}$ and $\theta$ is the oriented angle $\widehat{\mathbf{t}\boldsymbol{\delta}}$ (note that $\mathbf{t}(u)$ and $\boldsymbol{\delta}(u)$ are tangent vectors to $\boldsymbol{\sigma}$ at the point $\boldsymbol{\gamma}(u)$). Recall from Example 5.3.1 that, for $\boldsymbol{\sigma}$ to be regular, $\mathbf{t}$ and $\boldsymbol{\delta}$ must not be parallel, so $\sin\theta \neq 0$.

7.3.13 Suppose that a surface patch $\boldsymbol{\sigma}(u,v)$ has first fundamental form

$$du^2 + 2\cos\theta\, dudv + dv^2,$$

where $\theta$ is a smooth function of $(u,v)$ (cf. Exercise 6.1.5). Show that the geodesic curvatures $\kappa'_g$ and $\kappa''_g$ of the parameter curves $v =$ constant and $u =$ constant, respectively, are given by

$$\kappa'_g = -\theta_u, \quad \kappa''_g = \theta_v.$$

7.3.14 In the notation of Exercise 7.3.4, suppose that $F = 0$. Show that

$$\kappa_g = \sqrt{EG}(\dot{u}\ddot{v} - \ddot{u}\dot{v}) + \frac{1}{2}\left\{-\sqrt{\frac{G}{E}}(E_u\dot{u} + E_v\dot{v}) + \sqrt{\frac{E}{G}}(G_u\dot{u} + G_v\dot{v})\right\}\dot{u}\dot{v}$$
$$+ \frac{1}{2\sqrt{EG}}(G_u\dot{v} - E_v\dot{u}).$$

7.3.15 Continuing to assume that $F = 0$, deduce from the preceding exercise that the geodesic curvatures $\kappa'_g$ and $\kappa''_g$ of the parameter curves $v =$ constant and $u =$ constant on $\boldsymbol{\sigma}$ are

$$\kappa'_g = -\frac{E_v}{2E\sqrt{G}}, \quad \kappa''_g = \frac{G_u}{2G\sqrt{E}},$$

respectively. Hence prove *Liouville's formula*: if $\theta$ (which may depend on $(u, v)$) is the oriented angle $\widehat{\dot{\boldsymbol{\gamma}}\boldsymbol{\sigma}_u}$ between $\boldsymbol{\gamma}$ and the curves $v =$ constant, the geodesic curvature of $\boldsymbol{\gamma}$ is

$$\kappa_g = \dot{\theta} + \kappa'_g \cos\theta + \kappa''_g \sin\theta.$$

An analogue of Liouville's formula for the normal curvature is given in Theorem 8.2.4.

7.3.16 Let $\mathbf{p}$ be a point on a curve $\mathcal{C}$ on a surface $\mathcal{S}$, and let $\Pi$ be the tangent plane to $\mathcal{S}$ at $\mathbf{p}$. Let $\tilde{\mathcal{C}}$ be the curve obtained by projecting $\mathcal{C}$ orthogonally onto $\Pi$. Show that the curvature of the plane curve $\tilde{\mathcal{C}}$ at $\mathbf{p}$ is equal, up to sign, to the geodesic curvature of $\mathcal{C}$ at $\mathbf{p}$.

7.3.17 Show that the asymptotic curves on the surface

$$\boldsymbol{\sigma}(u, v) = \left(u, v, \frac{1}{2}(u^2 - v^2)\right)$$

are straight lines.

7.3.18 Let $\boldsymbol{\gamma}$ be a unit-speed curve and consider the ruled surface

$$\boldsymbol{\sigma}(u, v) = \boldsymbol{\gamma}(u) + v\mathbf{n}(u),$$

where $\mathbf{n}$ is the principal normal of $\boldsymbol{\gamma}$. Prove that $\boldsymbol{\gamma}$ is an asymptotic curve on $\boldsymbol{\sigma}$.

7.3.19 A surface is obtained by rotating the parabola $z^2 = 4x$ in the $xz$-plane around the $z$-axis (this is *not* a paraboloid). Show that the orthogonal projections of the asymptotic curves on the surface onto the $xy$-plane are logarithmic spirals (when suitably parametrized). (See Example 1.2.2.)

7.3.20 Let $\boldsymbol{\gamma}$ be a curve on a surface $\mathcal{S}$, and assume that $\mathcal{C}$ has nowhere vanishing curvature. Show that $\boldsymbol{\gamma}$ is asymptotic if and only if the osculating plane at every point $\mathbf{p}$ of $\boldsymbol{\gamma}$ is parallel to the tangent plane of $\mathcal{S}$ at $\mathbf{p}$.

7.3.21 Show that, if every curve on a surface is asymptotic, the surface is an open subset of a plane.

7.3.22 Let $\boldsymbol{\gamma}$ be a unit-speed curve on an oriented surface with curvature $\kappa > 0$. Let $\psi$ be the angle between $\ddot{\boldsymbol{\gamma}}$ and $\mathbf{N}$, and let $\mathbf{B} = \mathbf{t} \times \mathbf{N}$ (in the usual notation). Show that

$$\mathbf{N} = \mathbf{n}\cos\psi + \mathbf{b}\sin\psi, \quad \mathbf{B} = \mathbf{b}\cos\psi - \mathbf{n}\sin\psi.$$

Deduce that

$$\dot{\mathbf{t}} = \kappa_n\mathbf{N} - \kappa_g\mathbf{B}, \quad \dot{\mathbf{N}} = -\kappa_n\mathbf{t} + \tau_g\mathbf{B}, \quad \dot{\mathbf{B}} = \kappa_g\mathbf{t} - \tau_g\mathbf{N},$$

where $\tau_g = \tau + \dot{\psi}$ ($\tau_g$ is called the *geodesic torsion* of $\boldsymbol{\gamma}$).

7.3.23 Show that an asymptotic curve with nowhere vanishing curvature has torsion equal to its geodesic torsion (see the preceding exercise).

7.4.1 Let $\tilde{\boldsymbol{\gamma}}$ be a reparametrization of $\boldsymbol{\gamma}$, so that $\tilde{\boldsymbol{\gamma}}(t) = \boldsymbol{\gamma}(\varphi(t))$ for some smooth function $\varphi$ with $d\varphi/dt \neq 0$ for all values of $t$. If $\mathbf{v}$ is a tangent vector field along $\boldsymbol{\gamma}$, show that $\tilde{\mathbf{v}}(t) = \mathbf{v}(\varphi(t))$ is one along $\tilde{\boldsymbol{\gamma}}$. Prove that

$$\nabla_{\tilde{\boldsymbol{\gamma}}}\tilde{\mathbf{v}} = \frac{d\varphi}{dt}\nabla_{\boldsymbol{\gamma}}\mathbf{v},$$

and deduce that $\mathbf{v}$ is parallel along $\boldsymbol{\gamma}$ if and only if $\tilde{\mathbf{v}}$ is parallel along $\tilde{\boldsymbol{\gamma}}$.

7.4.2 Show that the parallel transport map $\Pi_{\boldsymbol{\gamma}}^{\mathbf{pq}} : T_{\mathbf{p}}\mathcal{S} \to T_{\mathbf{q}}\mathcal{S}$ is invertible. What is its inverse ?

7.4.3 Suppose that a triangle on the unit sphere whose sides are arcs of great circles has vertices $\mathbf{p}, \mathbf{q}, \mathbf{r}$. Let $\mathbf{v}_0$ be a non-zero tangent vector to the arc $\overline{\mathbf{pq}}$ through $\mathbf{p}$ and $\mathbf{q}$ at $\mathbf{p}$. Show that, if we parallel transport $\mathbf{v}_0$ along $\overline{\mathbf{pq}}$, then along $\overline{\mathbf{qr}}$ and then along $\overline{\mathbf{rp}}$, the result is to rotate $\mathbf{v}_0$ through an angle $2\pi - \mathcal{A}$, where $\mathcal{A}$ is the area of the triangle. For an analogous result see Theorem 13.6.4.

7.4.4 Calculate the Christoffel symbols when the first fundamental form is

$$du^2 + 2\cos\theta\, dudv + dv^2$$

for some smooth function $\theta(u, v)$ (Exercise 6.1.5).

7.4.5 Let $\boldsymbol{\sigma}(u, v)$ be a surface patch. Show that the following are equivalent:
(i) The parameter curves of $\boldsymbol{\sigma}(u, v)$ form a Chebyshev net (see Exercise 6.1.5).
(ii) The tangent vectors to the parameter curves $u =$ constant are parallel along each parameter curve $v =$ constant.
(iii) The tangent vectors to the parameter curves $v =$ constant are parallel along each parameter curve $u =$ constant.

7.4.6 Let $\theta = \widehat{\boldsymbol{\sigma}_u\boldsymbol{\sigma}_v}$ be the oriented angle between the parameter curves of a surface patch $\boldsymbol{\sigma}$ with first fundamental form $Edu^2 + 2Fdudv + Gdv^2$. Show that

$$\theta_u = -A\left(\frac{\Gamma_{11}^2}{E} + \frac{\Gamma_{12}^1}{G}\right), \quad \theta_v = -A\left(\frac{\Gamma_{12}^2}{E} + \frac{\Gamma_{22}^1}{G}\right),$$

where $A = \sqrt{EG - F^2}$.

7.4.7 With the notation in the preceding exercise, show that

$$\frac{A_u}{A} = \Gamma_{11}^1 + \Gamma_{12}^2, \quad \frac{A_v}{A} = \Gamma_{22}^2 + \Gamma_{12}^1.$$

# Chapter 8

8.1.1 Show that the Gaussian and mean curvatures of the surface $z = f(x, y)$, where $f$ is a smooth function, are

$$K = \frac{f_{xx}f_{yy} - f_{xy}^2}{(1 + f_x^2 + f_y^2)^2}, \quad H = \frac{(1 + f_y^2)f_{xx} - 2f_xf_yf_{xy} + (1 + f_x^2)f_{yy}}{2(1 + f_x^2 + f_y^2)^{3/2}}.$$

8.1.2 Calculate the Gaussian curvature of the helicoid and catenoid (Exercises 4.2.6 and 5.3.1).

8.1.3 Show that the Gaussian and mean curvatures of a surface $\mathcal{S}$ are smooth functions on $\mathcal{S}$.

8.1.4 In the notation of Example 8.1.5, show that if $\boldsymbol{\delta}$ is the principal normal $\mathbf{n}$ of $\boldsymbol{\gamma}$ or its binormal $\mathbf{b}$, then $K = 0$ if and only if $\boldsymbol{\gamma}$ is planar.

8.1.5 What is the effect on the Gaussian and mean curvatures of a surface $\mathcal{S}$ if we apply a dilation of $\mathbb{R}^3$ to $\mathcal{S}$?

8.1.6 Show that the Weingarten map $\mathcal{W}$ of a surface satisfies the quadratic equation

$$\mathcal{W}^2 - 2H\mathcal{W} + K = 0,$$

in the usual notation.

8.1.7 Show that the image of the Gauss map of a generalized cone is a *curve* on $S^2$, and deduce that the cone has zero Gaussian curvature.

8.1.8 Let $\boldsymbol{\sigma} : U \to \mathbb{R}^3$ be a patch of a surface $\mathcal{S}$. Show that the image under the Gauss map of the part $\boldsymbol{\sigma}(R)$ of $\mathcal{S}$ corresponding to a region $R \subseteq U$ has area

$$\int\!\!\int_R |K| d\mathcal{A}_{\boldsymbol{\sigma}},$$

where $K$ is the Gaussian curvature of $\mathcal{S}$.

8.1.9 Let $\mathcal{S}$ be the torus in Exercise 4.2.5. Describe the parts $\mathcal{S}^+$ and $\mathcal{S}^-$ of $\mathcal{S}$ where the Gaussian curvature $K$ of $\mathcal{S}$ is positive and negative, respectively. Show, without calculation, that

$$\int\int_{\mathcal{S}^+} K\, d\mathcal{A} = -\int\int_{\mathcal{S}^-} K\, d\mathcal{A} = 4\pi.$$

It follows that $\int\int_{\mathcal{S}} K\, d\mathcal{A} = 0$, a result that will be 'explained' in §13.4.

8.1.10 Let $\mathbf{w}(u,v)$ be a *smooth tangent vector field on a surface patch* $\boldsymbol{\sigma}(u,v)$. This means that

$$\mathbf{w}(u,v) = \alpha(u,v)\boldsymbol{\sigma}_u + \beta(u,v)\boldsymbol{\sigma}_v$$

where $\alpha$ and $\beta$ are smooth functions of $(u,v)$. Then, if $\boldsymbol{\gamma}(t) = \boldsymbol{\sigma}(u(t), v(t))$ is any curve on $\boldsymbol{\sigma}$, $\mathbf{w}$ gives rise to the tangent vector field $\mathbf{w}|_{\boldsymbol{\gamma}}(t) = \mathbf{w}(u(t), v(t))$ along $\boldsymbol{\gamma}$. Let $\nabla_u \mathbf{w}$ be the covariant derivative of $\mathbf{w}|_{\boldsymbol{\gamma}}$ along a parameter curve $v =$ constant, and define $\nabla_v \mathbf{w}$ similarly. (Note that if $\boldsymbol{\sigma}$ is the $uv$-plane, then $\nabla_u$ and $\nabla_v$ become $\partial/\partial u$ and $\partial/\partial v$). Show that

$$\nabla_v(\nabla_u \mathbf{w}) - \nabla_u(\nabla_v \mathbf{w}) = (\mathbf{w}_v.\mathbf{N})\mathbf{N}_u - (\mathbf{w}_u.\mathbf{N})\mathbf{N}_v,$$

where $\mathbf{N}$ is the unit normal of $\boldsymbol{\sigma}$. Deduce that, if $\lambda(u,v)$ is a smooth function of $(u,v)$, then

$$\nabla_v(\nabla_u(\lambda\mathbf{w})) - \nabla_u(\nabla_v(\lambda\mathbf{w})) = \lambda\left(\nabla_v(\nabla_u \mathbf{w}) - \nabla_u(\nabla_v \mathbf{w})\right).$$

Use Proposition 8.1.2 to show that

$$\nabla_v(\nabla_u \boldsymbol{\sigma}_u) - \nabla_u(\nabla_v \boldsymbol{\sigma}_u) = K(-F\boldsymbol{\sigma}_u + E\boldsymbol{\sigma}_v),$$

where

$$K = \frac{LN - M^2}{EG - F^2},$$

and find a similar expression for $\nabla_v(\nabla_u \boldsymbol{\sigma}_v) - \nabla_u(\nabla_v \boldsymbol{\sigma}_v)$. Deduce that

$$\nabla_v(\nabla_u \mathbf{w}) = \nabla_u(\nabla_v \mathbf{w})$$

for all tangent vector fields $\mathbf{w}$ if and only if $K = 0$ everywhere on the surface. (Note that this holds for the plane: $\mathbf{w}_{uv} = \mathbf{w}_{vu}$.) We shall see the significance of the condition $K = 0$ in §8.4.

8.1.11 Calculate the Gaussian and mean curvatures of the surface

$$\boldsymbol{\sigma}(u,v) = (u+v, u-v, uv)$$

at the point $(2, 0, 1)$.

8.1.12 Consider the quadric surface

$$\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 1,$$

where we can assume that the non-zero constants $a, b, c$ satisfy $a > 0$, $a > b > c$. Thus, if $c > 0$ we have an ellipsoid; if $b > 0 > c$ a hyperboloid of one sheet; and if $b < 0$ a hyperboloid of two sheets. Show that the Gaussian curvature at a point $\mathbf{p}$ of the quadric is

$$K = \frac{d^4}{abc},$$

where $d$ is the distance from $\mathbf{p}$ to $T_{\mathbf{p}}\mathcal{S}$.

Obtain a similar result for the paraboloid

$$\frac{x^2}{a} + \frac{y^2}{b} = 4z,$$

where $a > b$ and $a > 0$ (an elliptic paraboloid if $b > 0$, a hyperbolic paraboloid if $b < 0$).

8.1.13 Show that the Gaussian curvature of the surface $\mathcal{S}$ with Cartesian equation $xyz = 1$ is

$$K = 3(x^{-2} + y^{-2} + z^{-2})^{-2},$$

and calculate its mean curvature. Show that the maximum value of $K$ is attained at exactly four points which form the vertices of a regular tetrahedron.

8.1.14 A circle initially in the $xz$-plane tangent to the $z$-axis is rotated at constant angular velocity around the $z$-axis at the same time as its centre moves at constant speed parallel to the $z$-axis. Show that the surface generated has a parametrization

$$\boldsymbol{\sigma}(u, v) = (a(1 + \cos u)\cos v, a(1 + \cos u)\sin v, a \sin u + bv + c),$$

where $a$, $b$ and $c$ are constants. (Compare Exercise 4.2.6.)

Assume that $a = b$ and $c = 0$. Show that the Gaussian curvature of $\boldsymbol{\sigma}$ at a point a distance $d$ from the $z$-axis is

$$\frac{3d - 4a}{4a^3}.$$

8.1.15 Show that, if the Gaussian curvature $K$ of a ruled surface is constant, then $K = 0$. A complete description of such surfaces is given in §8.4.

8.1.16 Show that the Gaussian curvature of the tube of radius $a$ around a unit-speed curve $\boldsymbol{\gamma}$ (see Exercise 4.2.7) is

$$K = \frac{-\kappa \cos\theta}{a(1 - \kappa a \cos\theta)},$$

where $\kappa$ is the curvature of $\boldsymbol{\gamma}$ (we assume that $\kappa < a^{-1}$ at every point of $\boldsymbol{\gamma}$). Note that $K$ does not depend on the torsion of $\boldsymbol{\gamma}$.
Suppose now that $\boldsymbol{\gamma}$ is a closed curve of length $\ell$. Show that:
(i) $\int_0^\ell \int_0^{2\pi} K\, d\mathcal{A} = 0$, where $d\mathcal{A}$ is the element of area on the tube.
(ii) $\int_0^\ell \int_0^{2\pi} |K|\, d\mathcal{A} = 4\int_0^\ell \kappa(s) ds$, where $\kappa(s)$ is the curvature of $\boldsymbol{\gamma}$ at the point $\boldsymbol{\gamma}(s)$.
The explanation for (i) appears in §13.4 and the geometrical meaning of (ii) in Theorem 8.1.6.

8.1.17 Show that the Gaussian and mean curvatures are unchanged by applying a direct isometry of $\mathbb{R}^3$. What about an opposite isometry?

8.1.18 Show that an asymptotic curve on a surface $\mathcal{S}$ is perpendicular to its image under the Gauss map at the corresponding point.

8.1.19 Let $\boldsymbol{\gamma}$ be a curve on an oriented surface $\mathcal{S}$ with unit normal $\mathbf{N}$. Show that

$$\dot{\mathbf{N}}.\dot{\mathbf{N}} + 2H\dot{\mathbf{N}}.\dot{\boldsymbol{\gamma}} + K\dot{\boldsymbol{\gamma}}.\dot{\boldsymbol{\gamma}} = 0.$$

Deduce that, if $\boldsymbol{\gamma}$ is an asymptotic curve on $\mathcal{S}$, its torsion $\tau$ is related to the Gaussian curvature $K$ of $\mathcal{S}$ by $\tau^2 = -K$.

8.2.1 Calculate the principal curvatures of the helicoid and the catenoid, defined in Exercises 4.2.6 and 5.3.1, respectively.

8.2.2 A curve $\boldsymbol{\gamma}$ on a surface $\mathcal{S}$ is called a *line of curvature* if the tangent vector of $\boldsymbol{\gamma}$ is a principal vector of $\mathcal{S}$ at all points of $\boldsymbol{\gamma}$ (a 'line' of curvature need not be a straight line!). Show that $\boldsymbol{\gamma}$ is a line of curvature if and only if

$$\dot{\mathbf{N}} = -\lambda\dot{\boldsymbol{\gamma}},$$

for some scalar $\lambda$, where $\mathbf{N}$ is the standard unit normal of $\boldsymbol{\sigma}$, and that in this case the corresponding principal curvature is $\lambda$. (This is called *Rodrigues's formula.*)

8.2.3 Show that a curve $\boldsymbol{\gamma}(t) = \boldsymbol{\sigma}(u(t), v(t))$ on a surface patch $\boldsymbol{\sigma}$ is a line of curvature if and only if (in the usual notation)

$$(EM - FL)\dot{u}^2 + (EN - GL)\dot{u}\dot{v} + (FN - GM)\dot{v}^2 = 0.$$

Deduce that all parameter curves are lines of curvature if and only if either
(i) the second fundamental form of $\boldsymbol{\sigma}$ is proportional to its first fundamental form, or
(ii) $F = M = 0$.
For which surfaces does (i) hold? Show that the meridians and parallels of a surface of revolution are lines of curvature.

8.2.4 In the notation of Example 8.1.5, show that if $\boldsymbol{\gamma}$ is a curve on a surface $\mathcal{S}$ and $\boldsymbol{\delta}$ is the unit normal of $\mathcal{S}$, then $K = 0$ if and only if $\boldsymbol{\gamma}$ is a line of curvature of $\mathcal{S}$.

8.2.5 Suppose that two surfaces $\mathcal{S}_1$ and $\mathcal{S}_2$ intersect in a curve $\mathcal{C}$ that is a line of curvature of $\mathcal{S}_1$. Show that $\mathcal{C}$ is a line of curvature of $\mathcal{S}_2$ if and only if the angle between the tangent planes of $\mathcal{S}_1$ and $\mathcal{S}_2$ is constant along $\mathcal{C}$.

8.2.6 Let $\mathbf{\Sigma} : W \to \mathbb{R}^3$ be a smooth function defined on an open subset $W$ of $\mathbb{R}^3$ such that, for each fixed value of $u$ (resp. $v$, $w$), $\mathbf{\Sigma}(u, v, w)$ is a (regular) surface patch. Assume also that

$$\mathbf{\Sigma}_u.\mathbf{\Sigma}_v = \mathbf{\Sigma}_v.\mathbf{\Sigma}_w = \mathbf{\Sigma}_w.\mathbf{\Sigma}_u = 0. \tag{5}$$

This means that the three families of surfaces formed by fixing the values of $u$, $v$ or $w$ constitute a triply orthogonal system (see §5.5).
(i) Show that $\mathbf{\Sigma}_u.\mathbf{\Sigma}_{vw} = \mathbf{\Sigma}_v.\mathbf{\Sigma}_{uw} = \mathbf{\Sigma}_w.\mathbf{\Sigma}_{uv} = 0$.
(ii) Show that, for each of the surfaces in the triply orthogonal system, the matrices $\mathcal{F}_I$ and $\mathcal{F}_{II}$ are diagonal.
(iii) Deduce that the intersection of any surface from one family of the triply orthogonal system with any surface from another family is a line of curvature on both surfaces. (This is called *Dupin's Theorem*.)

8.2.7 Show that, if $p$, $q$ and $r$ are distinct positive numbers, there are exactly four umbilics on the ellipsoid

$$\frac{x^2}{p^2} + \frac{y^2}{q^2} + \frac{z^2}{r^2} = 1.$$

What happens if $p, q$ and $r$ are not distinct?

8.2.8 Show that the principal curvatures of a surface patch $\boldsymbol{\sigma} : U \to \mathbb{R}^3$ are smooth functions on $U$ provided that $\boldsymbol{\sigma}$ has no umbilics. Show also that the principal curvatures either stay the same or both change sign when $\boldsymbol{\sigma}$ is reparametrized.

8.2.9 Show that the principal curvatures of the surface

$$y \cos\frac{z}{a} = x \sin\frac{z}{a},$$

where $a$ is a non-zero constant, are

$$\pm\frac{a}{x^2 + y^2 + a^2}.$$

In particular, the mean curvature of the surface is zero.

8.2.10 Show that a point of a surface is an umbilic if and only if $H^2 = K$ at that point (in the notation of Definition 8.1.1). Deduce that a surface with $K < 0$ has no umbilics.

8.2.11 Show that the umbilics of the surface in Exercise 8.1.13 coincide with the points at which the Gaussian curvature of the surface attains its maximum value.

8.2.12 Suppose that a surface $\mathcal{S}$ has positive Gaussian curvature everywhere. Show that every curve on $\mathcal{S}$ has positive curvature everywhere (in particular, there are no straight line segments on $\mathcal{S}$).

8.2.13 Show that the principal curvatures of a surface $\mathcal{S}$ change sign when the orientation of $\mathcal{S}$ changes (i.e. when the unit normal of $\mathcal{S}$ changes sign), but that the principal vectors are unchanged.

8.2.14 Show that applying a direct isometry of $\mathbb{R}^3$ to a surface leaves the principal curvatures unchanged, but that an opposite isometry changes their sign.

8.2.15 Suppose that $m$ curves on a surface $\mathcal{S}$ all pass through a point $\mathbf{p}$ of $\mathcal{S}$ and that adjacent curves make equal angles $\pi/m$ with one another at $\mathbf{p}$. Show that the sum of the normal curvatures of the curves at $\mathbf{p}$ is equal to $mH$, where $H$ is the mean curvature of $\mathcal{S}$ at $\mathbf{p}$.

8.2.16 Find the lines of curvature on a tangent developable (§6.2).

8.2.17 In the notation of Exercise 8.2.4, assume that $\boldsymbol{\gamma}$ is a line of curvature of $\mathcal{S}$. Show that the ruled surface is
(i) a generalized cone if and only if the corresponding principal curvature is a non-zero constant along $\boldsymbol{\gamma}$;
(ii) a generalized cylinder if and only if the corresponding principal curvature is zero at all points of $\boldsymbol{\gamma}$.

8.2.18 Let $\boldsymbol{\gamma}$ be a curve on a surface $\mathcal{S}$. Show that $\boldsymbol{\gamma}$ is a line of curvature on $\mathcal{S}$ if and only if, at each point of $\boldsymbol{\gamma}$, the tangent vector of $\boldsymbol{\gamma}$ is parallel to that of the image of $\boldsymbol{\gamma}$ under the Gauss map of $\mathcal{S}$ at the corresponding point. Deduce that, if $\mathbf{p}$ is a point of a surface $\mathcal{S}$ that is not an umbilic, the Gauss map of $\mathcal{S}$ takes the two lines of curvature of $\mathcal{S}$ passing through $\mathbf{p}$ to orthogonal curves on $S^2$.

8.2.19 Show that if every curve on a (connected) surface $\mathcal{S}$ is a line of curvature, then $\mathcal{S}$ is an open subset of a plane or a sphere.

8.2.20 Show that a curve on a surface is a line of curvature if and only if its geodesic torsion vanishes everywhere (see Exercise 7.3.22).

8.2.21 Let $\boldsymbol{\gamma}$ be a line of curvature of a surface $\mathcal{S}$, and suppose that at each point of $\boldsymbol{\gamma}$ the osculating plane of $\boldsymbol{\gamma}$ makes the same angle with the tangent plane of $\mathcal{S}$. Show that $\boldsymbol{\gamma}$ is a plane curve.
Show conversely that, if a plane cuts a surface everywhere at the same angle, the intersection is a line of curvature on the surface.

8.2.22 Let $\boldsymbol{\gamma}$ be a curve on a surface with principal curvatures $\kappa_1$ and $\kappa_2$, and let $\theta$ be the angle between the tangent vector of $\boldsymbol{\gamma}$ and a non-zero principal vector corresponding to $\kappa_1$. Prove that the geodesic torsion (Exercise 7.3.22) of $\boldsymbol{\gamma}$ is given by

$$\tau_g = (\kappa_2 - \kappa_1)\sin\theta\cos\theta.$$

8.2.23 Show that, if there is an asymptotic curve passing through a point $\mathbf{p}$ of a surface $\mathcal{S}$, the principal curvatures $\kappa_1$ and $\kappa_2$ of $\mathcal{S}$ at $\mathbf{p}$ satisfy $\kappa_1\kappa_2 \leq 0$. Hence give another proof that the Gaussian curvature of a ruled surface is $\leq 0$ everywhere. Show, conversely, that if $\kappa_1\kappa_2 < 0$ everywhere there are exactly two asymptotic curves passing through each point of $\mathcal{S}$ and that the angle between them is

$$2\tan^{-1}\sqrt{-\frac{\kappa_1}{\kappa_2}}.$$

(This result will be generalized in Exercise 8.2.27.)
What if $\mathcal{S}$ is flat?

8.2.24 If $\mathbf{t}$ and $\tilde{\mathbf{t}}$ are tangent vectors at a point of a surface $\mathcal{S}$, one says that $\tilde{\mathbf{t}}$ *is conjugate to* $\mathbf{t}$ if

$$\langle\langle \mathbf{t}, \tilde{\mathbf{t}} \rangle\rangle = 0,$$

where $\langle\langle\ ,\ \rangle\rangle$ is the second fundamental form of $\mathcal{S}$. Show that:

(i) If $\tilde{\mathbf{t}}$ is conjugate to $\mathbf{t}$ then $\mathbf{t}$ is conjugate to $\tilde{\mathbf{t}}$.

(ii) If $\tilde{\mathbf{t}}_1$ and $\tilde{\mathbf{t}}_2$ are conjugate to $\mathbf{t}$, so is $\lambda_1\mathbf{t}_1 + \lambda_2\mathbf{t}_2$ for all $\lambda_1, \lambda_2 \in \mathbb{R}$.

8.2.25 Show that a curve on a surface is asymptotic if and only if its tangent vector is self-conjugate at every point of the curve.

8.2.26 Show that, if $\mathbf{t}_1$ and $\mathbf{t}_2$ are principal vectors corresponding to distinct principal curvatures, then $\mathbf{t}_1$ is conjugate to $\mathbf{t}_2$.

8.2.27 Let $\mathbf{t}_1$ and $\mathbf{t}_2$ be unit principal vectors at a point $\mathbf{p}$ of a surface $\mathcal{S}$ corresponding to principal curvatures $\kappa_1$ and $\kappa_2$. Let $\mathbf{t}$ and $\tilde{\mathbf{t}}$ be unit tangent vectors to $\mathcal{S}$ at $\mathbf{p}$ and let $\theta$ and $\tilde{\theta}$ be the oriented angles $\widehat{\mathbf{t}_1\mathbf{t}}$ and $\widehat{\mathbf{t}_1\tilde{\mathbf{t}}}$, respectively. Show that $\tilde{\mathbf{t}}$ is conjugate to $\mathbf{t}$ if and only if

$$\tan\theta\tan\tilde{\theta} = -\frac{\kappa_1}{\kappa_2}.$$

8.2.28 Let $\boldsymbol{\gamma}$ and $\tilde{\boldsymbol{\gamma}}$ be curves on a surface $\mathcal{S}$ that intersect at a point $\mathbf{p}$, and assume that the tangent vectors of $\boldsymbol{\gamma}$ and $\tilde{\boldsymbol{\gamma}}$ at $\mathbf{p}$ are conjugate. Show that, if $\kappa_n$ and $\tilde{\kappa}_n$ are the normal curvatures of $\boldsymbol{\gamma}$ and $\tilde{\boldsymbol{\gamma}}$ at $\mathbf{p}$,

$$\frac{1}{\kappa_n} + \frac{1}{\tilde{\kappa}_n} = \frac{1}{\kappa_1} + \frac{1}{\kappa_2},$$

where $\kappa_1$ and $\kappa_2$ are the principal curvatures of $\mathcal{S}$ at $\mathbf{p}$ (assumed to be non-zero).

8.2.29 Let $\mathbf{v}$ be a tangent vector field along a curve $\boldsymbol{\gamma}$ on a surface. Show that, if $\dot{\mathbf{v}}(t) = \mathbf{0}$ for some value of $t$, then $\mathbf{v}(t)$ is conjugate to $\dot{\boldsymbol{\gamma}}(t)$.

8.2.30 With the notation in Exercise 4.5.5, show that the first and second fundamental forms of $\mathcal{S}$ at a point $\mathbf{p}$ and of $\mathcal{S}^*$ at $F(\mathbf{p})$ are related by (in an obvious notation)

$$E^* = \frac{E}{\| \mathbf{p} \|^4}, \quad F^* = \frac{F}{\| \mathbf{p} \|^4}, \quad G^* = \frac{G}{\| \mathbf{p} \|^4},$$

$$L^* = -\frac{L}{\| \mathbf{p} \|^2} - 2\frac{E(\mathbf{p}.\mathbf{N})}{\| \mathbf{p} \|^4}, \ M^* = -\frac{M}{\| \mathbf{p} \|^2} - 2\frac{F(\mathbf{p}.\mathbf{N})}{\| \mathbf{p} \|^4}, \ N^* = -\frac{N}{\| \mathbf{p} \|^2} - 2\frac{G(\mathbf{p}.\mathbf{N})}{\| \mathbf{p} \|^4}$$

where $\mathbf{N}$ is the unit normal of $\mathcal{S}$. Deduce that:
(i) $F$ takes lines of curvature on $\mathcal{S}$ to lines of curvature on $\mathcal{S}^*$.
(ii) If $\kappa$ is a principal curvature of $\mathcal{S}$ at $\mathbf{p}$, then

$$\kappa^* = - \| \mathbf{p} \|^2 \kappa - 2(\mathbf{p}.\mathbf{N})$$

is one of $\mathcal{S}^*$ at $F(\mathbf{p})$.
(iii) $F$ takes umbilics on $\mathcal{S}$ to umbilics on $\mathcal{S}^*$.

8.2.31 The *third fundamental form* of an oriented surface $\mathcal{S}$ is defined by

$$\langle\langle\langle \mathbf{v}, \mathbf{w} \rangle\rangle\rangle = \langle \mathcal{W}(\mathbf{v}), \mathcal{W}(\mathbf{v}) \rangle.$$

It is obvious that $\langle\langle\langle \mathbf{v}, \mathbf{w} \rangle\rangle\rangle$ is a symmetric bilinear form. Show that, if $\boldsymbol{\sigma}(u, v)$ is a surface patch of $\mathcal{S}$, the matrix of $\langle\langle\langle\ ,\ \rangle\rangle\rangle$ with respect to the basis $\{\boldsymbol{\sigma}_u, \boldsymbol{\sigma}_v\}$ of the tangent plane is (in the usual notation) $\mathcal{F}_{III} = \mathcal{F}_{II}\mathcal{F}_I^{-1}\mathcal{F}_{II}$.

8.2.32 Suppose that every point on the surface of revolution

$$\boldsymbol{\sigma}(u, v) = (f(u) \cos v, f(u) \sin v, g(u))$$

is parabolic. Show that:
(i) The zeros of $\dot{g} = dg/du$ are isolated (i.e. if $\dot{g}(u_0) = 0$ there is an $\epsilon > 0$ such that $\dot{g}(t) \neq 0$ if $0 < |u - u_0| < \epsilon$).
(ii) If $\dot{g}$ is never zero, $\boldsymbol{\sigma}$ is an open subset of a circular cylinder or a circular cone.

8.2.33 Show that the umbilics on a graph surface $z = f(x, y)$ satisfy

$$z_{xx} = \lambda(1 + z_x^2), \quad z_{xy} = \lambda z_x z_y, \quad z_{yy} = \lambda(1 + z_y^2)$$

for some $\lambda$ (possibly depending on $x$ and $y$).

8.2.34 Show that applying an isometry of $\mathbb{R}^3$ to a surface takes umbilics, elliptic, hyperbolic, parabolic and planar points of a surface to points of the same type.

8.2.35 Show that there are
(i) exactly four umbilics on a hyperboloid of two sheets;
(ii) exactly two umbilics on an elliptic paraboloid;
(iii) no umbilics on a hyperboloid of one sheet or a hyperbolic paraboloid.

8.2.36 Show that, for the surface $\mathcal{S}$ in Exercise 8.1.13, the four points at which the Gaussian curvature attains its maximum value are exactly the umbilics of $\mathcal{S}$.

8.2.37 Show that $\mathbf{p} = (1, 1, 1)$ is a planar point of the surface $\mathcal{S}$ with Cartesian equation

$$x^2y^2 + y^2z^2 + z^2x^2 = 3.$$

Deduce that the eight points $(\pm1, \pm1, \pm1)$ are all planar points of $\mathcal{S}$. (It can be shown that these are all the planar points of $\mathcal{S}$.)

Determine the shape of $\mathcal{S}$ near $\mathbf{p}$ as follows.

(i) Show that the vector $\mathbf{n} = (1, 1, 1)$ is normal to $\mathcal{S}$ at $\mathbf{p}$ and that the vectors $\mathbf{t}_1 = (1, -1, 0)$ and $\mathbf{t}_2 = (0, 1, -1)$ are tangent to $\mathcal{S}$ at $\mathbf{p}$.

(ii) For any point $(x, y, z) \in \mathbb{R}^3$ near $\mathbf{p}$ we can write

$$(x, y, z) = \mathbf{p} + X\mathbf{t}_1 + Y\mathbf{t}_2 + Z\mathbf{n}$$

for some small quantities $X, Y, Z$ depending on $x, y, z$. From the discussion at the end of §8.2, we know that, near $\mathbf{p}$, $Z$ is equal to a cubic polynomial in $X$ and $Y$, if we neglect terms of higher order. Show that, if we neglect such terms, then

$$2Z = XY(X - Y)$$

near $\mathbf{p}$.

(iii) Deduce that, near $\mathbf{p}$, $\mathcal{S}$ has the shape of a monkey saddle.

8.3.1 Show tha:

(i) Setting $w = e^{-u}$ gives a reparametrization $\boldsymbol{\sigma}_1(v, w)$ of the pseudosphere with first fundamental form

$$\frac{dv^2 + dw^2}{w^2}$$

(called the *upper half-plane model*).

(ii) Setting

$$V = \frac{v^2 + w^2 - 1}{v^2 + (w + 1)^2}, \quad W = \frac{-2v}{v^2 + (w + 1)^2}$$

defines a reparametrization $\boldsymbol{\sigma}_2(V, W)$ of the pseudosphere with first fundamental form

$$\frac{4(dV^2 + dW^2)}{(1 - V^2 - W^2)^2}$$

(called the *Poincaré disc model*: the region $w > 0$ of the $vw$-plane corresponds to the disc $V^2 + W^2 < 1$ in the $VW$-plane).

(iii) Setting

$$\bar{V} = \frac{2V}{V^2 + W^2 + 1}, \quad \bar{W} = \frac{2W}{V^2 + W^2 + 1}$$

defines a reparametrization $\boldsymbol{\sigma}_2(\bar{V}, \bar{W})$ of the pseudosphere with first fundamental form

$$\frac{(1-\bar{W}^2)d\bar{V}^2 + 2\bar{V}\bar{W}d\bar{V}d\bar{W} + (1-\bar{V}^2)d\bar{W}^2}{(1-\bar{V}^2-\bar{W}^2)^2}$$

(called the *Beltrami-Klein model*: the region $w > 0$ of the $vw$-plane again corresponds to the disc $\bar{V}^2 + \bar{W}^2 < 1$ in the $\bar{V}\bar{W}$-plane).
These models are discussed in much more detail in Chapter 11.

8.3.2 For the pseudosphere:
(i) Calculate the length of a parallel.
(ii) Calculate its total area.
(iii) Calculate the principal curvatures.
(iv) Show that all points are hyperbolic.

8.3.3 Let $\mathcal{S}$ be a surface of revolution with axis the $z$-axis, and let its profile curve be a unit-speed curve $\boldsymbol{\gamma}(u)$ in the $xz$-plane. Suppose that $\boldsymbol{\gamma}$ intersects the $z$-axis at right angles when $u = \pm\pi/2$, but does not intersect the $z$-axis when $-\pi/2 < u < \pi/2$. Prove that, if the Gaussian curvature $K$ of $\mathcal{S}$ is constant, that constant is equal to one and $\mathcal{S}$ is the unit sphere.

8.4.1 Let $\mathbf{p}$ be a hyperbolic point of a surface $\mathcal{S}$ (see §8.2). Show that there is a patch of $\mathcal{S}$ containing $\mathbf{p}$ whose parameter curves are asymptotic curves (see Exercise 7.3.6). Show that the second fundamental form of such a patch is of the form $2M\,dudv$.

8.4.2 Find a reparametrization of the hyperbolic paraboloid

$$\boldsymbol{\sigma}(u, v) = (u+v, u-v, uv)$$

in terms of parameters $(s, t)$ such that the lines of curvature are the parameter curves $s =$ constant and $t =$ constant.

8.4.3 Let $\boldsymbol{\gamma}$ be a curve on a surface $\mathcal{S}$, and let $\tilde{\mathcal{S}}$ be the ruled surface formed by the straight lines passing through points $\mathbf{p}$ of the curve that are tangent to $\mathcal{S}$ at $\mathbf{p}$ and intersect the curve orthogonally. Show that $\tilde{\mathcal{S}}$ is flat if and only if $\boldsymbol{\gamma}$ is a line of curvature of $\mathcal{S}$.

8.5.1 Suppose that the first fundamental form of a surface patch $\boldsymbol{\sigma}(u, v)$ is of the form $E(du^2 + dv^2)$. Prove that $\boldsymbol{\sigma}_{uu} + \boldsymbol{\sigma}_{vv}$ is perpendicular to $\boldsymbol{\sigma}_u$ and $\boldsymbol{\sigma}_v$. Deduce that the mean curvature $H = 0$ everywhere if and only if the Laplacian

$$\boldsymbol{\sigma}_{uu} + \boldsymbol{\sigma}_{vv} = \mathbf{0}.$$

Show that the surface patch

$$\boldsymbol{\sigma}(u, v) = \left(u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + u^2v, u^2 - v^2\right)$$

has $H = 0$ everywhere. (A picture of this surface can be found in §12.2.)

8.5.2 Prove that $H = 0$ for the surface

$$z = \ln\left(\frac{\cos y}{\cos x}\right).$$

(A picture of this surface can also be found in §12.2.)

8.5.3 Let $\boldsymbol{\sigma}(u, v)$ be a surface with first and second fundamental forms $Edu^2 + Gdv^2$ and $Ldu^2 + Ndv^2$, respectively (cf. Proposition 8.4.1). Define

$$\boldsymbol{\Sigma}(u, v, w) = \boldsymbol{\sigma}(u, v) + w\mathbf{N}(u, v),$$

where $\mathbf{N}$ is the standard unit normal of $\boldsymbol{\sigma}$. Show that the three families of surfaces obtained by fixing the values of $u$, $v$ or $w$ in $\boldsymbol{\Sigma}$ form a triply orthogonal system (see §5.5). The surfaces $w =$ constant are parallel surfaces of $\boldsymbol{\sigma}$. Show that the surfaces $u =$ constant and $v =$ constant are flat ruled surfaces.

8.5.4 Show that a ruled surface which has constant non-zero mean curvature is a circular cylinder.

8.5.5 Show that the lines of curvature on a parallel surface of a surface $\mathcal{S}$ correspond to those of $\mathcal{S}$, and that their tangents at corresponding points are parallel.

8.5.6 Suppose that two surface $\mathcal{S}$ and $\tilde{\mathcal{S}}$ have the same normal lines. Show that $\tilde{\mathcal{S}}$ is a parallel surface of $\mathcal{S}$ (cf. Exercise 2.3.15).

# Chapter 9

9.1.1 Describe four different geodesics on the hyperboloid of one sheet

$$x^2 + y^2 - z^2 = 1$$

passing through the point $(1, 0, 0)$.

9.1.2 A (regular) curve $\boldsymbol{\gamma}$ with nowhere vanishing curvature on a surface $\mathcal{S}$ is called a *pre-geodesic* on $\mathcal{S}$ if some reparametrization of $\boldsymbol{\gamma}$ is a geodesic on $\mathcal{S}$ (recall that a reparametrization of a geodesic is not usually a geodesic). Show that:
(i) A curve $\boldsymbol{\gamma}$ is a pre-geodesic if and only if $\ddot{\boldsymbol{\gamma}}.(\mathbf{N} \times \dot{\boldsymbol{\gamma}}) = 0$ everywhere on $\boldsymbol{\gamma}$ (in the notation of the proof of Proposition 9.1.3).
(ii) Any reparametrization of a pre-geodesic is a pre-geodesic.
(iii) Any constant speed reparametrization of a pre-geodesic is a geodesic.
(iv) A pre-geodesic is a geodesic if and only if it has constant speed.

9.1.3 Consider the tube of radius $a > 0$ around a unit-speed curve $\boldsymbol{\gamma}$ in $\mathbb{R}^3$ defined in Exercise 4.2.7:

$$\boldsymbol{\sigma}(s, \theta) = \boldsymbol{\gamma}(s) + a(\cos\theta\,\mathbf{n}(s) + \sin\theta\,\mathbf{b}(s)).$$

Show that the parameter curves on the tube obtained by fixing the value of $s$ are circular geodesics on $\boldsymbol{\sigma}$.

9.1.4 Let $\boldsymbol{\gamma}(t)$ be a geodesic on an ellipsoid $\mathcal{S}$ (see Theorem 5.2.2(i)). Let $2R(t)$ be the length of the diameter of $\mathcal{S}$ parallel to $\dot{\boldsymbol{\gamma}}(t)$, and let $S(t)$ be the distance from the centre of $\mathcal{S}$ to the tangent plane $T_{\boldsymbol{\gamma}(t)}\mathcal{S}$. Show that the curvature of $\boldsymbol{\gamma}$ is $S(t)/R(t)^2$, and that the product $R(t)S(t)$ is independent of $t$.

9.1.5 Show that a geodesic with nowhere vanishing curvature is a plane curve if and only if it is a line of curvature.

9.1.6 Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two surfaces that intersect in a curve $\mathcal{C}$, and let $\boldsymbol{\gamma}$ be a unit-speed parametrization of $\mathcal{C}$.
(i) Show that if $\boldsymbol{\gamma}$ is a geodesic on *both* $\mathcal{S}_1$ and $\mathcal{S}_2$ and if the curvature of $\boldsymbol{\gamma}$ is nowhere zero, then $\mathcal{S}_1$ ad $\mathcal{S}_2$ touch along $\boldsymbol{\gamma}$ (i.e. they have the same tangent plane at each point of $\mathcal{C}$). Give an example of this situation.
(ii) Show that if $\mathcal{S}_1$ and $\mathcal{S}_2$ intersect orthogonally at each point of $\mathcal{C}$, then $\boldsymbol{\gamma}$ is a geodesic on $\mathcal{S}_1$ if and only if $\dot{\mathbf{N}}_2$ is parallel to $\mathbf{N}_1$ at each point of $\mathcal{C}$ (where $\mathbf{N}_1$ and $\mathbf{N}_2$ are unit normals of $\mathcal{S}_1$ and $\mathcal{S}_2$). Show also that, in this case, $\boldsymbol{\gamma}$ is a geodesic on *both* $\mathcal{S}_1$ and $\mathcal{S}_2$ if and only if $\mathcal{C}$ is part of a straight line.

9.1.7 Show that the ellipsoid

$$\frac{x^2}{p^2} + \frac{y^2}{q^2} + \frac{z^2}{r^2} = 1$$

always has at least three closed geodesics.

9.1.8 Find six geodesics on (each connected piece of) the surface in Exercise 8.2.37.

9.1.9 Suppose that the tangent vector to a geodesic $\boldsymbol{\gamma}$ with nowhere vanishing curvature on a surface $\mathcal{S}$ makes a fixed angle with a fixed non-zero vector $\mathbf{a}$. Show that, at every point of $\boldsymbol{\gamma}$, the vector $\mathbf{a}$ is tangent to $\mathcal{S}$.

9.1.10 Deduce from Exercise 7.3.3 that great circles are the only circles on a sphere that are geodesics.

9.1.11 Let $\boldsymbol{\gamma}$ be a unit-speed curve on a surface $\mathcal{S}$. Show that $\boldsymbol{\gamma}$ is a geodesic on $\mathcal{S}$ if and only if, at every point $\mathbf{p}$ of $\boldsymbol{\gamma}$, the osculating plane of $\boldsymbol{\gamma}$ at $\mathbf{p}$ is perpendicular to $T_{\mathbf{p}}\mathcal{S}$. Dedcuce that, if a geodesic $\boldsymbol{\gamma}$ on $\mathcal{S}$ is the intersection of $\mathcal{S}$ with a plane, then $\boldsymbol{\gamma}$ is a normal section of $\mathcal{S}$ (this is a converse of Proposition 9.1.6).

9.1.12 Show that if a curve on a surface is both a geodesic and an asymptotic curve, then it is part of a straight line.

9.1.13 Show that a unit-speed curve $\boldsymbol{\gamma}$ with nowhere vanishing curvature is a geodesic on the ruled surface

$$\boldsymbol{\sigma}(u,v) = \boldsymbol{\gamma}(u) + v\boldsymbol{\delta}(u),$$

where $\boldsymbol{\delta}$ is a smooth function of $u$, if and only if $\boldsymbol{\delta}(u)$ is perpendicular to the principal normal of $\boldsymbol{\gamma}$ at $\boldsymbol{\gamma}(u)$ for all values of $u$.

9.1.14 Let $\mathbf{\Gamma}$ be a unit-speed geodesic on a surface $\mathcal{S}$. Show that the torsion of $\mathbf{\Gamma}$ at a point $\mathbf{p}$ is

$$\tau = \dot{\mathbf{\Gamma}}.(\mathbf{N} \times \dot{\mathbf{N}}),$$

where $\mathbf{N}$ is the standard unit normal of any surface patch containing $\mathbf{p}$. Suppose now that a curve $\boldsymbol{\gamma}$ on $\boldsymbol{\sigma}$ touches $\mathbf{\Gamma}$ at a point $\mathbf{p}$. Show that the torsion of $\mathbf{\Gamma}$ at $\mathbf{p}$ is equal to the geodesic torsion of $\boldsymbol{\gamma}$ at $\mathbf{p}$ (Exercise 7.3.22). Deduce, in particular, that the torsion of $\mathbf{\Gamma}$ is equal to its geodesic torsion.

9.1.15 Show that the torsion of a unit-speed asymptotic curve on a surface is given by the same formula as the torsion of a geodesic in the preceding exercise. Deduce that if a geodesic touches an asymptotic curve at a point $\mathbf{p}$, the two curves have the same torsion at $\mathbf{p}$.

9.1.16 Show that, if a geodesic touches a line of curvature at a point $\mathbf{p}$, the torsion of the geodesic vanishes at $\mathbf{p}$.

9.1.17 Show that:
(i) The torsion of a geodesic vanishes at an umbilic.
(ii) Two geodesics that intersect at right angles at a point $\mathbf{p}$ have torsions at $\mathbf{p}$ that are equal in magnitude but opposite in sign.
(iii) The curvature $\kappa$ and torsion $\tau$ of a geodesic are related by

$$\tau^2 = -(\kappa - \kappa_1)(\kappa - \kappa_2) \quad \text{or} \quad -(\kappa + \kappa_1)(\kappa + \kappa_2),$$

where $\kappa_1$ and $\kappa_2$ are the principal curvatures.
(iv) If the surface is flat then, up to a sign, $\tau = \kappa \tan\theta$ or $\kappa \cot\theta$, where $\theta$ is the angle between the geodesic and one of the lines of curvature.

9.1.18 Let $\boldsymbol{\gamma}$ be a curve on a ruled surface $\mathcal{S}$ that intersects each of the rulings of the surface. Show that, if $\boldsymbol{\gamma}$ has any two of the following properties, it has all three:
(i) $\boldsymbol{\gamma}$ is a pre-geodesic on $\mathcal{S}$.
(ii) $\boldsymbol{\gamma}$ is the line of striction of $\mathcal{S}$ (see Exercise 5.3.4).
(iii) $\boldsymbol{\gamma}$ cuts the rulings of $\mathcal{S}$ at a constant angle.

9.1.19 Suppose that every geodesic on a (connected) surface is a *plane* curve. Show that the surface is an open subset of a plane or a sphere.

9.1.20 Suppose that a geodesic $\boldsymbol{\gamma}$ on a surface $\mathcal{S}$ lies on a sphere with centre $\mathbf{c}$. Show that the curvature of $\boldsymbol{\gamma}$ at a point $\mathbf{p}$ is the reciprocal of the length of the perpendicular from $\mathbf{c}$ to the plane passing through $\mathbf{p}$ parallel to $T_{\mathbf{p}}\mathcal{S}$.

9.2.1 Show that, if $\mathbf{p}$ and $\mathbf{q}$ are distinct points of a circular cylinder, there are either two or infinitely-many geodesics on the cylinder with endpoints $\mathbf{p}$ and $\mathbf{q}$ (and which do not otherwise pass through $\mathbf{p}$ or $\mathbf{q}$). Which pairs $\mathbf{p}$, $\mathbf{q}$ have the former property?

9.2.2 Use Corollary 9.2.8 to find all the geodesics on a circular cone.

9.2.3 Find the geodesics on the unit cylinder by solving the geodesic equations.

9.2.4 Consider the following three properties that a curve $\boldsymbol{\gamma}$ on a surface may have:
(i) $\boldsymbol{\gamma}$ has constant speed.
(ii) $\boldsymbol{\gamma}$ satisfies the first of the geodesic equations (9.2).
(iii) $\boldsymbol{\gamma}$ satisfies the second of the geodesic equations (9.2).
Show that (ii) and (iii) together imply (i). Show also that if (i) holds and if $\boldsymbol{\gamma}$ is not a parameter curve, then (ii) and (iii) are equivalent.

9.2.5 Let $\boldsymbol{\gamma}(t)$ be a unit-speed curve on the helicoid

$$\boldsymbol{\sigma}(u, v) = (u \cos v, u \sin v, v)$$

(Exercise 4.2.6). Show that

$$\dot{u}^2 + (1 + u^2)\dot{v}^2 = 1$$

(a dot denotes $d/dt$). Show also that, if $\boldsymbol{\gamma}$ is a geodesic on $\boldsymbol{\sigma}$, then

$$\dot{v} = \frac{a}{1 + u^2},$$

where $a$ is a constant. Find the geodesics corresponding to $a = 0$ and $a = 1$. Suppose that a geodesic $\boldsymbol{\gamma}$ on $\boldsymbol{\sigma}$ intersects a ruling at a point $\mathbf{p}$ a distance $D > 0$ from the $z$-axis, and that the angle between $\boldsymbol{\gamma}$ and the ruling at $\mathbf{p}$ is $\alpha$, where $0 < \alpha < \pi/2$. Show that the geodesic intersects the $z$-axis if $D > \cot \alpha$, but that if $D < \cot \alpha$ its smallest distance from the $z$-axis is $\sqrt{D^2 \sin^2 \alpha - \cos^2 \alpha}$. Find the equation of the geodesic if $D = \cot \alpha$.

9.2.6 Verify directly that the differential equations in Proposition 9.2.3 are equivalent to the geodesic equations in Theorem 9.2.1.

9.2.7 Use Corollary 9.2.7 to show that the geodesics on a generalized cylinder are exactly those constant-speed curves on the cylinder whose tangent vector makes a constant angle with the rulings of the cylinder.

9.2.8 Show that (in the usual notation) a parameter curve $v$ = constant is a pre-geodesic on a surface patch $\boldsymbol{\sigma}$ if and only if

$$EE_v + FE_u = 2EF_u.$$

9.2.9 Suppose that a surface patch $\boldsymbol{\sigma}$ has first fundamental form

$$(1 + u^2)du^2 - 2uvdudv + (1 + v^2)dv^2.$$

Show that the curves on $\boldsymbol{\sigma}$ corresponding to the straight lines $u + v$ = constant in the $uv$-plane are pre-geodesics on $\boldsymbol{\sigma}$.

9.2.10 Suppose that the first fundamental form of a surface $\boldsymbol{\sigma}$ is of the form

$$du^2 + Gdv^2,$$

and that the curves on the surface corresponding to the straight lines $v/u =$ constant are all pre-geodesics. Prove that

$$(2u^2 + Gv^2)G_u + uvG_v = 0,$$

and verify that this equation has a solution of the form

$$G(u, v) = \frac{f(u)}{v^2}.$$

9.2.11 Suppose that the coefficients $E, F, G$ of the first fundamental form of a surface patch $\boldsymbol{\sigma}(u, v)$ depend only on $u$. Show that, along any geodesic on $\boldsymbol{\sigma}$, either $u =$ constant or

$$\frac{dv}{du} = -\frac{F}{G} \pm \frac{\Omega\sqrt{EG - F^2}}{G\sqrt{G - \Omega^2}},$$

where $\Omega$ is a constant.

9.2.12 Let $\boldsymbol{\gamma}$ be a curve and let $\mathcal{S}$ be the ruled surface generated by its binormals (see Exercise 6.1.9). Suppose that $\boldsymbol{\Gamma}$ is a geodesic on $\mathcal{S}$ that intersects $\boldsymbol{\gamma}$. Show that:
(i) If the torsion of $\boldsymbol{\gamma}$ is a non-zero constant, then $\boldsymbol{\Gamma}$ is contained between two rulings of $\mathcal{S}$.
(ii) If $\boldsymbol{\gamma}$ is a plane curve, then $\boldsymbol{\Gamma}$ is contained between two rulings only if $\boldsymbol{\gamma}$ and $\boldsymbol{\Gamma}$ intersect perpendicularly, in which case $\boldsymbol{\Gamma}$ is one of the rulings of $\mathcal{S}$.

9.2.13 A *Liouville surface* is a surface patch $\boldsymbol{\sigma}$ whose first fundamental form is of the form

$$(U + V)(Pdu^2 + Qdv^2),$$

where $U$ and $P$ are functions of $u$ only and $V$ and $Q$ are functions of $v$ only. Show that, if $\boldsymbol{\gamma}$ is a geodesic on $\boldsymbol{\sigma}$, then along $\boldsymbol{\gamma}$,

$$U \sin^2\theta - V \cos^2\theta = \text{constant},$$

where $\theta$ is the angle between $\boldsymbol{\gamma}$ and the parameter curves $v =$ constant.

9.2.14 Verify that

$$\boldsymbol{\sigma}(u, v) = \left(\sqrt{\frac{a(a+u)(a+v)}{(a-b)(a-c)}}, \sqrt{\frac{b(b+u)(b+v)}{(b-a)(b-c)}}, \sqrt{\frac{c(c+u)(c+v)}{(c-a)(c-b)}}\right)$$

is a parametrization of the quadric surface

$$\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 1,$$

where $a, b, c$ are distinct. Determine its first fundamental form and deduce that, along any geodesic on the surface,

$$u \sin^2 \theta + v \cos^2 \theta = \text{ constant},$$

where $\theta$ is the angle at which the geodesic intersects the parameter curves $v =$ constant.

9.3.1 There is another way to see that all the meridians, and all the parallels corresponding to stationary points of $f$, are geodesics on a surface of revolution. What is it?

9.3.2 Describe qualitatively the geodesics on
(i) a spheroid, obtained by rotating an ellipse around one of its axes;
(ii) a torus (Exercise 4.2.5).

9.3.3 Show that a geodesic on the pseudosphere with non-zero angular momentum $\Omega$ intersects itself if and only if $\Omega < (1 + \pi^2)^{-1/2}$. How many self-intersections are there in that case?

9.3.4 Show that if we reparametrize the pseudosphere as in Exercise 8.3.1(ii), the geodesics on the pseudosphere correspond to segments of straight lines and circles in the parameter plane that intersect the boundary of the disc orthogonally. Deduce that, in the parametrization of Exercise 8.3.1(iii), the geodesics correspond to segments of *straight lines* in the parameter plane. We shall see in §10.4 that there are very few surfaces that have parametrizations with this property.

9.3.5 Suppose that a surface of revolution has the property that *every* parallel is a geodesic. What kind of surface is it?

9.3.6 Show that, along any geodesic on the catenoid (Exercise 5.3.1) that is not a parallel,

$$\frac{dv}{du} = \pm \frac{\Omega}{\sqrt{\cosh^2 u - \Omega^2}},$$

where $\Omega$ is a constant.

9.3.7 Show that, if every geodesic on a surface of revolution $\mathcal{S}$ intersects the meridians at a constant angle (possibly different angles for different geodesics), then $\mathcal{S}$ is a circular cylinder.

9.3.8 Deduce from Exercise 4.1.6 and Clairaut's theorem that the geodesics on the unit cylinder are straight lines, circles and helices.

9.3.9 Consider the surface of revolution

$$\boldsymbol{\sigma}(u,v) = \left(k\cos u\cos v, k\cos u\sin v, \int_0^u \sqrt{1-k^2\sin^2\theta}\,d\theta\right)$$

where $k$ is a non-zero constant. (This is a surface of constant Gaussian curvature equal to 1 and is a sphere if $k = 1$ - see §8.3.) Prove that the geodesic which passes through the point $(k,0,0)$ and makes an angle $\alpha$ with the parallel through this point is given by

$$\tan u = \pm\tan\alpha\sin kv.$$

Find the maximum height above the $xy$-plane attained by this geodesic.

9.3.10 Let $f : \tilde{\boldsymbol{\sigma}}(v,w) \mapsto \tilde{\boldsymbol{\sigma}}(\tilde{v},\tilde{w})$ be an isometry of the pseudosphere, where the parametrization $\tilde{\boldsymbol{\sigma}}$ is that defined in §9.3.
(i) Show that $f$ takes meridians to meridians, and deduce that $\tilde{v}$ does not depend on $w$.
(ii) Deduce that $f$ takes parallels to parallels.
(iii) Deduce from (ii) and Exercise 8.3.2 that $\tilde{w} = w$.
(iv) Show that $f$ is a rotation about the axis of the pseudosphere or a reflection in a plane containing the axis of rotation.

9.4.1 The geodesics on a circular (half) cone were determined in Exercise 9.2.2. Interpreting 'line' as 'geodesic', which of the following (true) statements in plane Euclidean geometry are true for the cone?
(i) There is a line passing through any two points.
(ii) There is a unique line passing through any two distinct points.
(iii) Any two distinct lines intersect in at most one point.
(iv) There are lines that do not intersect each other.
(v) Any line can be continued indefinitely.
(vi) A line defines the shortest distance between any two of its points.
(vii) A line cannot intersect itself transversely (i.e. with two non-parallel tangent vectors at the point of intersection).

9.4.2 Show that the long great circle arc on $S^2$ joining the points $\mathbf{p} = (1,0,0)$ and $\mathbf{q} = (0,1,0)$ is not even a *local* minimum of the length function $\mathcal{L}$ (see the remarks following the proof of Theorem 9.4.1).

9.4.3 Construct a smooth function with the properties in (9.20) in the following steps:
(i) Show that, for all integers $n$ (positive and negative), $t^n e^{-1/t^2}$ tends to 0 as $t$ tends to 0.
(ii) Deduce from (i) that the function

$$\theta(t) = \begin{cases} e^{-1/t^2} & \text{if } t \geq 0, \\ 0 & \text{if } t \leq 0 \end{cases}$$

is smooth everywhere.
(iii) Show that the function

$$\psi(t) = \theta(1+t)\theta(1-t)$$

is smooth everywhere, that $\psi(t) > 0$ if $-1 < t < 1$, and that $\psi(t) = 0$ otherwise.
(iv) Show that the function

$$\phi(t) = \psi\left(\frac{t - t_0}{\eta}\right)$$

has the properties we want.

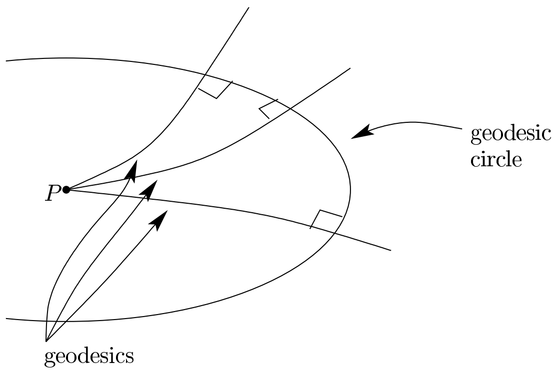
9.4.4 Repeat Exercise 9.4.1 for a circular cylinder and a sphere.

9.5.1 Let $P$ be a point of a surface $\mathcal{S}$ and let $\mathbf{v}$ be a unit tangent vector to $\mathcal{S}$ at $P$. Let $\boldsymbol{\gamma}^\theta(r)$ be the unit-speed geodesic on $\mathcal{S}$ passing through $P$ when $r = 0$ and such that the oriented angle $\widehat{\mathbf{v}\,\frac{d\boldsymbol{\gamma}^\theta}{dr}} = \theta$. It can be shown that $\boldsymbol{\sigma}(r,\theta) = \boldsymbol{\gamma}^\theta(r)$ is smooth for $-\epsilon < r < \epsilon$ and all values of $\theta$, where $\epsilon$ is some positive number, and that it is an allowable surface patch for $\mathcal{S}$ defined for $0 < r < \epsilon$ and for $\theta$ in any open interval of length $\leq 2\pi$. This is called a *geodesic polar patch* on $\mathcal{S}$.
Show that, if $0 < R < \epsilon$,

$$\int_0^R \left\| \frac{d\boldsymbol{\gamma}^\theta}{dr} \right\|^2 dr = R.$$

By differentiating both sides with respect to $\theta$, prove that

$$\boldsymbol{\sigma}_r . \boldsymbol{\sigma}_\theta = 0.$$




This is called *Gauss's Lemma* – geometrically, it means that the parameter curve $r = R$, called the *geodesic circle* with centre $P$ and radius $R$, is perpendicular to each of its radii, i.e. the geodesics passing through $P$. Deduce that the first fundamental form of $\boldsymbol{\sigma}$ is

$$dr^2 + G(r,\theta)d\theta^2,$$

for some smooth function $G(r, \theta)$.

9.5.2 Let $P$ and $Q$ be two points on a surface $\mathcal{S}$, and assume that there is a geodesic polar patch with centre $P$ as in Exercise 9.5.1 that also contains $Q$; suppose that $Q$ is the point $\boldsymbol{\sigma}(R, \alpha)$, where $0 < R < \epsilon$, $0 \leq \alpha < 2\pi$. Show in the following steps that the geodesic $\boldsymbol{\gamma}^\alpha(t) = \boldsymbol{\sigma}(t, \alpha)$ is (up to reparametrization) the unique *shortest* curve on $\mathcal{S}$ joining $P$ and $Q$.
(i) Let $\boldsymbol{\gamma}(t) = \boldsymbol{\sigma}(f(t), g(t))$ be any curve in the patch $\boldsymbol{\sigma}$ joining $P$ and $Q$. We assume that $\boldsymbol{\gamma}$ passes through $P$ when $t = 0$ and through $Q$ when $t = R$ (this can always be achieved by a suitable reparametrization). Show that the length of the part of $\boldsymbol{\gamma}$ between $P$ and $Q$ is $\geq R$, and that $R$ is the length of the part of $\boldsymbol{\gamma}^\alpha$ between $P$ and $Q$.
(ii) Show that, if $\boldsymbol{\gamma}$ is *any* curve on $\mathcal{S}$ joining $P$ and $Q$ (not necessarily staying inside the patch $\boldsymbol{\sigma}$), the length of the part of $\boldsymbol{\gamma}$ between $P$ and $Q$ is $\geq R$.
(iii) Show that, if the part of a curve $\boldsymbol{\gamma}$ on $\mathcal{S}$ joining $P$ to $Q$ has length $R$, then $\boldsymbol{\gamma}$ is a reparametrization of $\boldsymbol{\gamma}^\alpha$.

9.5.3 Suppose that every geodesic circle with centre $P$ in the surface patch in Exercise 9.5.1 has constant geodesic curvature (possibly different constants for different circles). Prove that $G$ is of the form $G(r, \theta) = f(r)g(\theta)$ for some smooth functions $f$ and $g$.

# Chapter 10

10.1.1 A surface patch has first and second fundamental forms

$$\cos^2 v \, du^2 + dv^2 \quad \text{and} \quad -\cos^2 v \, du^2 - dv^2,$$

respectively. Show that the surface is an open subset of a sphere of radius one. Write down a parametrization of $S^2$ with these first and second fundamental forms.

10.1.2 Show that there is no surface patch whose first and second fundamental forms are

$$du^2 + \cos^2 u \, dv^2 \quad \text{and} \quad \cos^2 u \, du^2 + dv^2,$$

respectively.

10.1.3 Suppose that a surface patch $\boldsymbol{\sigma}(v, w)$ has first and second fundamental forms

$$\frac{dv^2 + dw^2}{w^2} \quad \text{and} \quad L dv^2 + N dw^2,$$

respectively, where $w > 0$. Prove that $L$ and $N$ do not depend on $v$, that $LN = -1/w^4$ and that

$$L w^5 \frac{dL}{dw} = 1 - L^2 w^4.$$

Solve this equation for $L$ and deduce that $\boldsymbol{\sigma}$ cannot be defined in the whole of the half-plane $w > 0$. Compare the discussion of the pseudosphere in Example 9.3.3.

10.1.4 Suppose that the first and second fundamental forms of a surface patch are $Edu^2 + Gdv^2$ and $Ldu^2 + Ndv^2$, respectively. Show that the Codazzi–Mainardi equations reduce to

$$L_v = \frac{1}{2}E_v\left(\frac{L}{E} + \frac{N}{G}\right), \quad N_u = \frac{1}{2}G_u\left(\frac{L}{E} + \frac{N}{G}\right).$$

Deduce that the principal curvatures $\kappa_1 = L/E$ and $\kappa_2 = N/G$ satisfy the equations

$$(\kappa_1)_v = \frac{E_v}{2E}(\kappa_2 - \kappa_1), \quad (\kappa_2)_u = \frac{G_u}{2G}(\kappa_1 - \kappa_2).$$

10.1.5 What are the necessary and sufficient conditions for *constants* $E, F, G, L, M, N$ to be the coefficients of the first and second fundamental forms of a surface patch $\boldsymbol{\sigma}(u, v)$?
Assuming that these conditions are satisfied, show that there is a reparametrization of $\boldsymbol{\sigma}$ of the form

$$\tilde{u} = au + bv, \qquad \tilde{v} = cu + dv,$$

where $a, b, c, d$ are constants, such that the first and second fundamental forms become

$$d\tilde{u}^2 + d\tilde{v}^2 \qquad \text{and} \qquad \kappa d\tilde{u}^2,$$

respectively, where $\kappa$ is a constant. Deduce that the surface is an open subset of a plane or a circular cylinder.

10.1.6 Suppose that a surface patch has first and second fundamental forms $Edu^2+Gdv^2$ and $2dudv$, respectively. Show that:
(i) $E/G$ is a constant.
(ii) By a suitable reparametrization we can arrange that this constant is equal to 1.
(iii) If $E = G$ then

$$\frac{\partial^2(\ln E)}{\partial u^2} + \frac{\partial^2(\ln E)}{\partial v^2} = \frac{2}{E}.$$

10.1.7 Show that, if the parameter curves of a surface patch are asymptotic curves,

$$\frac{M_u}{M} = \Gamma_{11}^1 - \Gamma_{12}^2, \quad \frac{M_v}{M} = \Gamma_{22}^2 - \Gamma_{12}^1.$$

10.1.8 Suppose that a surface $\mathcal{S}$ has no umbilics and that one of its principal curvatures is a non-zero constant $\kappa$. Let $\mathbf{p} \in \mathcal{S}$.

(i) Show that there is a patch $\boldsymbol{\sigma}(u,v)$ of $\mathcal{S}$ containing $\mathbf{p}$ which has first and second fundamental forms

$$du^2 + Gdv^2 \quad \text{and} \quad \kappa du^2 + Ndv^2,$$

for some smooth functions $G$ and $N$ of $(u,v)$.

(ii) Calculate the Christoffel symbols of $\boldsymbol{\sigma}$.

(iii) Show that the parameter curves $v = \text{constant}$ are circles of radius $r = 1/|\kappa|$.

(iv) Show that

$$\boldsymbol{\sigma}_{uu} + r^2\boldsymbol{\sigma}$$

is independent of $u$, and deduce that

$$\boldsymbol{\sigma}(u,v) = \boldsymbol{\gamma}(v) + r(\mathbf{c}(v)\cos\frac{u}{r} + \mathbf{d}(v)\sin\frac{u}{r}),$$

for some curve $\boldsymbol{\gamma}(v)$, where $\mathbf{c}(v)$ and $\mathbf{d}(v)$ are perpendicular unit vectors for all values of $v$.

(v) Show that $\mathbf{c}$ and $\mathbf{d}$ are perpendicular to $d\boldsymbol{\gamma}/dv$ and deduce that $\boldsymbol{\sigma}$ is a reparametrization of the tube of radius $r$ around $\boldsymbol{\gamma}$.

10.1.9 Let $\boldsymbol{\Sigma}(u,v,w)$ be a parametrization of a triply orthogonal system as in §5.5. Prove that, if $p = \parallel \boldsymbol{\Sigma}_u \parallel$, $q = \parallel \boldsymbol{\Sigma}_v \parallel$, $r = \parallel \boldsymbol{\Sigma}_w \parallel$, then

$$\begin{aligned}
\boldsymbol{\Sigma}_{uu} &= \frac{p_u}{p}\boldsymbol{\Sigma}_u - \frac{pp_v}{q^2}\boldsymbol{\Sigma}_v - \frac{pp_w}{r^2}\boldsymbol{\Sigma}_w,\\
\boldsymbol{\Sigma}_{vv} &= \frac{q_v}{q}\boldsymbol{\Sigma}_v - \frac{qq_w}{r^2}\boldsymbol{\Sigma}_w - \frac{qq_u}{p^2}\boldsymbol{\Sigma}_u,\\
\boldsymbol{\Sigma}_{ww} &= \frac{r_w}{r}\boldsymbol{\Sigma}_w - \frac{rr_u}{p^2}\boldsymbol{\Sigma}_u - \frac{rr_v}{q^2}\boldsymbol{\Sigma}_v,\\
\boldsymbol{\Sigma}_{vw} &= \frac{q_w}{q}\boldsymbol{\Sigma}_v + \frac{r_v}{r}\boldsymbol{\Sigma}_w,\\
\boldsymbol{\Sigma}_{wu} &= \frac{r_u}{r}\boldsymbol{\Sigma}_w + \frac{p_w}{p}\boldsymbol{\Sigma}_u,\\
\boldsymbol{\Sigma}_{uv} &= \frac{p_v}{p}\boldsymbol{\Sigma}_u + \frac{q_u}{q}\boldsymbol{\Sigma}_v.
\end{aligned}$$

Deduce *Lamé's relations*:

$$\left(\frac{r_v}{q}\right)_v + \left(\frac{q_w}{r}\right)_w + \frac{q_u r_u}{p^2} = 1,$$
$$\left(\frac{p_w}{r}\right)_w + \left(\frac{r_u}{p}\right)_u + \frac{r_v p_v}{q^2} = 1,$$
$$\left(\frac{q_u}{p}\right)_u + \left(\frac{p_v}{q}\right)_v + \frac{p_w q_w}{r^2} = 1,$$
$$p_{vw} = \frac{p_v q_w}{q} + \frac{p_w r_v}{r},$$
$$q_{wu} = \frac{q_w r_u}{r} + \frac{q_u p_w}{p},$$
$$r_{uv} = \frac{r_u p_v}{p} + \frac{r_v q_u}{q}.$$

It can be shown, conversely, that if $p, q, r$ are smooth functions of $u, v, w$ satisfying Lamé's relations, there is a triply orthogonal system $\mathbf{\Sigma}(u, v, w)$, uniquely determined up to an isometry of $\mathbb{R}^3$, such that $p = \parallel \mathbf{\Sigma}_u \parallel$, $q = \parallel \mathbf{\Sigma}_v \parallel$, $r = \parallel \mathbf{\Sigma}_w \parallel$.

10.2.1 Show that if a surface patch has first fundamental form $e^\lambda(du^2 + dv^2)$, where $\lambda$ is a smooth function of $u$ and $v$, its Gaussian curvature $K$ satisfies

$$\Delta\lambda + 2Ke^\lambda = 0,$$

where $\Delta$ denotes the Laplacian $\partial^2/\partial u^2 + \partial^2/\partial v^2$.

10.2.2 With the notation of Exercise 9.5.1, define $u = r\cos\theta$, $v = r\sin\theta$, and let $\tilde{\boldsymbol{\sigma}}(u, v)$ be the corresponding reparametrization of $\boldsymbol{\sigma}$. It can be shown that $\tilde{\boldsymbol{\sigma}}$ is an allowable surface patch for $\mathcal{S}$ defined on the open set $u^2 + v^2 < \epsilon^2$. (Note that this is not quite obvious because $\boldsymbol{\sigma}$ is not allowable when $r = 0$.)
(i) Show that the first fundamental form of $\tilde{\boldsymbol{\sigma}}$ is $\tilde{E}du^2 + 2\tilde{F}dudv + \tilde{G}dv^2$, where

$$\tilde{E} = \frac{u^2}{r^2} + \frac{Gv^2}{r^4}, \quad \tilde{F} = \left(1 - \frac{G}{r^2}\right)\frac{uv}{r^2}, \quad \tilde{G} = \frac{v^2}{r^2} + \frac{Gu^2}{r^4}.$$

(ii) Show that $u^2(\tilde{E} - 1) = v^2(\tilde{G} - 1)$, and by considering the Taylor expansions of $\tilde{E}$ and $\tilde{G}$ about $u = v = 0$, deduce that

$$G(r, \theta) = r^2 + kr^4 + \text{remainder}$$

for some constant $k$, where remainder$/r^4$ tends to zero as $r$ tends to zero.
(iii) Show that $k = -K(\mathbf{p})/3$, where $K(\mathbf{p})$ is the value of the Gaussian curvature of $\mathcal{S}$ at $\mathbf{p}$.

10.2.3 With the notation of Exercises 9.5.1 and 10.2.2, show that:

(i) The circumference of the geodesic circle with centre $\mathbf{p}$ and radius $R$ is

$$C_R = 2\pi R\left(1 - \frac{K(\mathbf{p})}{6}R^2 + \text{remainder}\right),$$

where remainder$/R^2$ tends to zero as $R$ tends to zero.

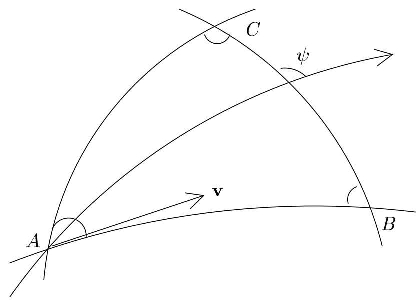
(ii) The area inside the geodesic circle in (i) is

$$A_R = \pi R^2\left(1 - \frac{K(\mathbf{p})}{12}R^2 + \text{remainder}\right),$$

where the remainder satisfies the same condition as in (i).

Verify that these formulas are consistent with those in spherical geometry obtained in Exercise 6.5.3.

10.2.4 Let $A, B$ and $C$ be the vertices of a triangle $\mathcal{T}$ on a surface $\mathcal{S}$ whose sides are arcs of geodesics, and let $\alpha, \beta$ and $\gamma$ be its internal angles (so that $\alpha$ is the angle at $A$, etc.). Assume that the triangle is contained in a geodesic patch $\boldsymbol{\sigma}$ as in Exercise 9.5.1 with $P = A$. Thus, with the notation in that exercise, if we take $\mathbf{v}$ to be tangent at $A$ to the side passing through $A$ and $B$, then the sides meeting at $A$ are the parameter curves $\theta = 0$ and $\theta = \alpha$, and the remaining side can be parametrized by $\boldsymbol{\gamma}(\theta) = \boldsymbol{\sigma}(f(\theta), \theta)$ for some smooth function $f$ and $0 \leq \theta \leq \alpha$.




(i) Use the geodesic equations (9.2) to show that

$$f'' - \frac{f'\lambda'}{\lambda^2} = \frac{1}{2}\frac{\partial G}{\partial r},$$

where a dash denotes $d/d\theta$ and $\lambda = \parallel \boldsymbol{\gamma}' \parallel$.

(ii) Show that, if $\psi(\theta)$ is the angle between $\boldsymbol{\sigma}_r$ and the tangent vector to the side opposite $A$ at $\boldsymbol{\gamma}(\theta)$, then

$$\psi'(\theta) = -\frac{\partial\sqrt{G}}{\partial r}(f(\theta), \theta).$$

(iii) Show that, if $K$ is the Gaussian curvature of $\mathcal{S}$,

$$\int\int_{\mathcal{T}} K\, d\mathcal{A}_{\boldsymbol{\sigma}} = \alpha + \beta + \gamma - \pi.$$

This result will be generalized in Corollary 13.2.3.

10.2.5 Show that the Gaussian curvature of the Möbius band in Example 4.5.3 is equal to $-1/4$ everywhere along its median circle. Deduce that this Möbius band *cannot* be constructed by taking a strip of paper and joining the ends together with a half-twist. (The analytic description of the 'cut and paste' Möbius band is more complicated than the version in Example 4.5.3.)

10.2.6 Show that the only isometries from the catenoid to itself are products of rotations around its axis, reflections in planes containing the axis, and reflection in the plane containing the waist of the catenoid.

10.2.7 A surface has first fundamental form

$$v^m du^2 + u^n dv^2$$

for some integers $m, n$. For which value(s) of the pair $(m, n)$ is this surface flat? Show directly that, in each case in which the surface is flat, it is locally isometric to a plane. (This is, of course, an immediate consequence of the results of §8.4.)

10.2.8 A surface patch $\boldsymbol{\sigma}$ has first fundamental form

$$du^2 + 2\cos\theta du dv + dv^2,$$

where $\theta$ is a smooth function of $(u, v)$ (Exercise 6.1.5). Show that the Gaussian curvature of $\boldsymbol{\sigma}$ is

$$K = -\frac{\theta_{uv}}{\sin\theta}.$$

Verify the Gauss equations (Proposition 10.1.2).

10.2.9 Show that there is no isometry between any region of a sphere and any region of a generalized cylinder or a generalized cone.

10.2.10 Consider the surface patches

$$\boldsymbol{\sigma}(u, v) = (u\cos v, u\sin v, \ln u), \quad \tilde{\boldsymbol{\sigma}}(u, v) = (u\cos v, u\sin v, v).$$

Prove that the Gaussian curvature of $\boldsymbol{\sigma}$ at $\boldsymbol{\sigma}(u, v)$ is the same as that of $\tilde{\boldsymbol{\sigma}}$ at $\tilde{\boldsymbol{\sigma}}(u, v)$, but that the map from $\boldsymbol{\sigma}$ to $\tilde{\boldsymbol{\sigma}}$ which takes $\boldsymbol{\sigma}(u, v)$ to $\tilde{\boldsymbol{\sigma}}(u, v)$ is *not* an isometry. Prove that, in fact, there is *no* isometry from $\boldsymbol{\sigma}$ to $\tilde{\boldsymbol{\sigma}}$.

10.2.11 Show that the only isometries of the torus in Example 4.2.5 are the maps $\boldsymbol{\sigma}(\theta, \varphi) \mapsto \boldsymbol{\sigma}(\pm\theta, \pm\varphi + \alpha)$, where $\alpha$ is any constant (and any combination of signs is allowed). Thus, the isometries are composites of reflection in the coordinate planes and rotations about the $z$-axis.

10.2.12 Show that, if the parameter curves of a surface are pre-geodesics that intersect orthogonally, the surface is flat. Is this still true without the assumption of orthogonality?

10.3.1 Show that a compact surface with Gaussian curvature $> 0$ everywhere and constant *mean* curvature is a sphere.

10.3.2 Show that the solution of the sine-Gordon equation corresponding to the pseudosphere constructed in §8.3 is

$$\theta(u, v) = 2\tan^{-1}(\sinh(u - v + c)),$$

where $c$ is a constant.

10.3.3 Let $\boldsymbol{\sigma}(u, v)$ be a surface patch of constant Gaussian curvature $-1$ such that the parameter curves form a Chebyshev net, as in Exercise 6.1.5. Let $\mathcal{Q}$ be a quadrilateral whose sides are parameter curves, and let $\alpha_1, \alpha_2, \alpha_3$ and $\alpha_4$ be its interior angles. Show that the area inside $\mathcal{Q}$ is

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 - 2\pi.$$

10.4.1 Show that a local diffeomorphism between surfaces that takes unit-speed geodesics to unit-speed geodesics must be a local isometry.

10.4.2 Show that a local diffeomorphism between surfaces that is the composite of a dilation and a local isometry takes geodesics to geodesics. Is the converse of this statement true?

10.4.3 This exercise shows that a geodesic local diffeomorphism $F$ from a surface $\mathcal{S}$ to a surface $\tilde{\mathcal{S}}$ that is also conformal is the composite of a dilation and a local isometry.
(i) Let $\mathbf{p} \in \mathcal{S}$ and let $\boldsymbol{\sigma}$ be a geodesic patch containing $\mathbf{p}$ as in Proposition 9.5.1, with first fundamental form $du^2 + Gdv^2$. Show that $\tilde{\boldsymbol{\sigma}} = F \circ \boldsymbol{\sigma}$ is a patch of $\tilde{\boldsymbol{\sigma}}$ containing $F(\mathbf{p})$ with first fundamental form $\lambda(du^2 + Gdv^2)$ for some smooth function $\lambda(u, v)$.
(ii) Show that the parameter curves $v =$ constant are pre-geodesics on $\tilde{\boldsymbol{\sigma}}$ and deduce that $\lambda$ is independent of $v$.
(iii) Show that if $\boldsymbol{\gamma}$ is a geodesic on $\boldsymbol{\sigma}$ and $\theta$ is the oriented angle between $\boldsymbol{\gamma}$ and the parameter curves $v =$ constant,

$$\frac{d\theta}{dv} + \frac{G_u}{2\sqrt{G}} = 0. \tag{10.19}$$

(iv) Show that

$$\frac{d\theta}{dv} + \frac{(\lambda G)_u}{2\lambda\sqrt{G}} = 0. \tag{10.20}$$

(v) Deduce from Eqs. (10.19) and (10.20) that $\lambda$ is constant.
(vi) Show that $F : \boldsymbol{\sigma} \to \tilde{\boldsymbol{\sigma}}$ is the composite of a dilation and a local isometry.

# Chapter 11

11.1.1 Show that, if $l$ is a half-line geodesic in $\mathcal{H}$ and $a$ is a point not on $l$, there are infinitely-many hyperbolic lines passing through $a$ that do not intersect $l$.

11.1.2 Complete the proof of Proposition 11.1.4 by dealing with the case in which the hyperbolic line passing through $a$ and $b$ is a half-line.

11.1.3 Show that for any $a \in \mathcal{H}$ there is a unique hyperbolic line passing through $a$ that intersects the hyperbolic line $l$ given by $v = 0$ perpendicularly. If $b$ is the point of intersection, one calls $d_{\mathcal{H}}(a, b)$ the *hyperbolic distance of $a$ from $l$.*

11.1.4 The *hyperbolic circle* $\mathcal{C}_{a,R}$ with centre $a \in \mathcal{H}$ and radius $R > 0$ is the set of points of $\mathcal{H}$ which are a hyperbolic distance $R$ from $a$:

$$\mathcal{C}_{a,R} = \{z \in \mathcal{H} \,|\, d_{\mathcal{H}}(z, a) = R\}.$$

Show that $\mathcal{C}_{a,R}$ is a Euclidean circle.
Show that the Euclidean centre of $\mathcal{C}_{ic,R}$, where $c > 0$, is $ib$ and that its Euclidean radius is $r$, where

$$c = \sqrt{b^2 - r^2}, \quad R = \frac{1}{2}\ln\frac{b+r}{b-r}.$$

Deduce that the hyperbolic length of the circumference of $\mathcal{C}_{ic,R}$ is $2\pi \sinh R$ and that the hyperbolic area inside it is $2\pi(\cosh R - 1)$. Note that these do not depend on $c$; in fact, it follows from the results of the next section that the circumference and area of $\mathcal{C}_{a,R}$ depend only on $R$ and not on $a$ (see the remarks preceding Theorem 11.2.4).
Compare these formulas with the case of a spherical circle in Exercise 6.5.3, and verify that they are consistent with Exercise 10.2.3.

11.1.5 Let $l$ be a half-line geodesic in $\mathcal{H}$. Show that, for any $R > 0$, the set of points that are a distance $R$ from $l$ is the union of two half-lines passing through the origin. Note that these half-lines are *not* geodesics. This contrasts with the situation in Euclidean geometry, in which the set of points at a fixed distance from a straight line is a pair of straight lines.

11.1.6 Which region in $\mathcal{H}$ corresponds to the pseudosphere with the meridian $v = \pi$ removed (in the parametrization used in §11.1)?

11.2.1 Show that if $a, b \in \mathcal{H}$, the hyperbolic distance $d_{\mathcal{H}}(a, b)$ is the length of the shortest curve in $\mathcal{H}$ joining $a$ and $b$.

11.2.2 Show that, if $l$ is any hyperbolic line in $\mathcal{H}$ and $a$ is a point not on $l$, there are infinitely-many hyperbolic lines passing through $a$ that do not intersect $l$.

11.2.3 Let $a$ be a point of $\mathcal{H}$ that is not on a hyperbolic line $l$. Show that there is a unique hyperbolic line $m$ passing through $a$ that intersects $l$ perpendicularly. If $b$ is the point of intersection of $l$ and $m$, and $c$ is any other point of $l$, prove that

$$d_{\mathcal{H}}(a, b) < d_{\mathcal{H}}(a, c).$$

Thus, $b$ is the unique point of $l$ that is closest to $a$.

11.2.4 This exercise and the next determine all the isometries of $\mathcal{H}$.
(i) Let $F$ be an isometry of $\mathcal{H}$ that fixes each point of the imaginary axis $l$ and each point of the semicircle geodesic $m$ of centre the origin and radius 1. Show that $F$ is the identity map.
(ii) Let $F$ be an isometry of $\mathcal{H}$ such that $F(l) = l$ and $F(m) = m$, where $l$ and $m$ are as in (i). Prove that $F$ is the identity map, the reflection $R_0$, the inversion $\mathcal{I}_{0,1}$ or the composite $\mathcal{I}_{0,1} \circ R_0$ (in the notation at the beginning of §11.2).
(iii) Show that every isometry of $\mathcal{H}$ is a composite of elementary isometries.
(iv) Show that every isometry of $\mathcal{H}$ is a composite of reflections and inversions in lines and circles perpendicular to the real axis.

11.2.5 A Möbius transformation (see Appendix 2) is said to be *real* if it is of the form

$$M(z) = \frac{az + b}{cz + d},$$

where $a, b, c, d \in \mathbb{R}$. Show that:
(i) Any composite of real Möbius transformations is a real Möbius transformation, and the inverse of any real Möbius transformation is a real Möbius transformation.
(ii) The Möbius transformations that take $\mathcal{H}$ to itself are exactly the real Möbius transformations such that $ad - bc > 0$.
(iii) Every real Möbius tranformation is a composite of elementary isometries of $\mathcal{H}$, and hence is an isometry of $\mathcal{H}$.
(iv) If $J(z) = -\bar{z}$ and $M$ is a real Möbius transformation, $M \circ J$ is an isometry of $\mathcal{H}$.
(v) If we call an isometry of type (iii) or (iv) a *Möbius isometry*, any composite of Möbius isometries is a Möbius isometry;
(vi) every isometry of $\mathcal{H}$ is a Möbius isometry.

11.2.6 Show that, if $a, b, c$ are three points of $\mathcal{H}$ that do not lie on the same geodesic, then

$$d_{\mathcal{H}}(a, b) < d_{\mathcal{H}}(a, c) + d_{\mathcal{H}}(c, b).$$

11.2.7 Let $l$ be a semicircle geodesic in $\mathcal{H}$ that intersects the real axis at points $a$ and $b$. Show that, for any $d > 0$, the set of points of $\mathcal{H}$ that are a hyperbolic distance $d$ from $l$ is the union of two circular arcs (called *equidistant curves*) passing through $a$ and $b$, but that these are not geodesics unless $d = 0$. Note that in Euclidean geometry equidistant curves are straight lines (i.e. geodesics).

11.2.8 Suppose that a triangle in $\mathcal{H}$ with vertices $a, b, c$ is such that $d_{\mathcal{H}}(a, b) = d_{\mathcal{H}}(a, c)$ (the triangle is 'isosceles') and let $\alpha$ be the angle at $a$. Prove that there is a function $f(\alpha)$ such that, if $d$ is the mid-point of the side joining $b, c$, then $d_{\mathcal{H}}(a, d) < f(\alpha)$. (The point is that this upper bound is independent of the lengths of the sides of the triangle passing through $a$, which are not bounded.)

11.2.9 Suppose that two triangles $T$ and $T'$ in $\mathcal{H}$ with vertices $a, b, c$ and $a', b', c'$ are such that
(i) the angle of $T$ at $a$ is equal to that of $T'$ at $a'$,
(ii) $d_{\mathcal{H}}(a, b) = d_{\mathcal{H}}(a', b')$, and
(iii) $d_{\mathcal{H}}(a, c) = d_{\mathcal{H}}(a', c')$.
Prove that $T$ and $T'$ are congruent.

11.2.10 Show that the set of points that are the same hyperbolic distance from two fixed points of $\mathcal{H}$ is a geodesic.

11.3.1 Prove Proposition 11.3.4.

11.3.2 Let $l$ and $m$ be hyperbolic lines in $\mathcal{D}_P$ that intersect at right angles. Prove that there is an isometry of $\mathcal{D}_P$ that takes $l$ to the real axis and $m$ to the imaginary axis. How many such isometries are there?

11.3.3 Show that the Möbius transformations that take $\mathcal{D}_P$ to itself are those of the form

$$z \mapsto \frac{az + b}{\bar{b}z + \bar{a}}, \quad |a| > |b|.$$

11.3.4 Show that the isometries of $\mathcal{D}_P$ are the transformations of the following two types:

$$z \mapsto \frac{az + b}{\bar{b}z + \bar{a}}, \qquad z \mapsto \frac{a\bar{z} + b}{\bar{b}\bar{z} + \bar{a}},$$

where $a$ and $b$ are complex numbers such that $|a| > |b|$. Note that this and the preceding exercise show that the isometries of $\mathcal{D}_P$ are exactly the Möbius and conjugate-Möbius transformations that take $\mathcal{D}_P$ to itself.

11.3.5 Prove that every isometry of $\mathcal{D}_P$ is the composite of finitely-many isometries of the two types in Proposition 11.3.3.

11.3.6 Consider a hyperbolic triangle with vertices $a, b, c$, sides of length $A, B, C$ and angles $\alpha, \beta, \gamma$ (so that $A$ is the length of the side opposite $a$ and $\alpha$ is the angle at $a$, etc.). Prove the *hyperbolic sine rule*

$$\frac{\sin \alpha}{\sinh A} = \frac{\sin \beta}{\sinh B} = \frac{\sin \gamma}{\sinh C}.$$

11.3.7 With the notation in the preceding exercise, suppose that $\gamma = \pi/2$. Prove that:
(i) $\cos\alpha = \frac{\sinh B \cosh A}{\sinh C}$.
(ii) $\cosh A = \frac{\cos\alpha}{\sin\beta}$.
(iii) $\sinh A = \frac{\tanh B}{\tan\beta}$.

11.3.8 With the notation in Exercise 11.3.6, prove that

$$\cosh A = \frac{\cos\alpha + \cos\beta\cos\gamma}{\sin\beta\sin\gamma}.$$

This is the formula we promised at the end of §11.2 for the lengths of the sides of a hyperbolic triangle in terms of its angles.

11.3.9 Show that if $\mathbb{R}^2$ is provided with the first fundamental form

$$\frac{4(du^2 + dv^2)}{(1 + u^2 + v^2)^2},$$

the stereographic projection map $\Pi : S^2 \backslash \{\text{north pole}\} \to \mathbb{R}^2$ defined in Example 6.3.5 is an isometry. Note the similarity between this formula and that in Proposition 11.3.2: the plane with this first fundamental form provides a 'model' for the sphere in the same way as the half-plane with the first fundamental form in Proposition 11.3.2 is a 'model' for the pseudosphere.

11.3.10 Show that Proposition 11.2.3 holds as stated in $\mathcal{D}_P$.

11.3.11 Let $n$ be an integer $\geq 3$. Show that, for any angle $\alpha$ such that $0 < \alpha < (n-2)\pi/n$, there is a regular hyperbolic $n$-gon with interior angles equal to $\alpha$. Show that each side of such an $n$-gon has length $A$, where

$$\cosh\frac{A}{2} = \frac{\cos\frac{\pi}{n}}{\sin\frac{\alpha}{2}}.$$

11.3.12 A *Saccheri quadrilateral* is a quadrilateral with geodesic sides such that two opposite sides have equal length $A$ and intersect a third side of length $B$ at right angles. If $C$ is the length of the fourth side, prove that

$$\cosh C = \cosh^2 A \cosh B - \sinh^2 A,$$

and determine the other two angles of the quadrilateral.

11.3.13 Prove that the set of points in $\mathcal{D}_P$ that are equidistant from two geodesics $l$ and $m$ that intersect at a point $a \in \mathcal{D}_P$ is the union of two hyperbolic lines that bisect the angles between $l$ and $m$. Deduce that (as in Euclidean geometry) the geodesics that bisect the internal angles of a hyperbolic triangle meet in a single point.

11.3.14 Let $a \in \mathcal{D}_P$ be a point on a geodesic $l$, let $b$ be a point on the geodesic $m$ that intersects $l$ perpendicularly at $a$, and let $c$ be one of the points in which $m$ intersects $\mathcal{C}$. Let $\mathcal{C}_b$ be the hyperbolic circle with centre $b$ that touches $l$ at $a$. Show that, as $b$ recedes from $a$ along $m$ towards $c$, the circles $\mathcal{C}_b$ approach a limiting curve (called a *horocycle*) which is a Euclidean circle touching $\mathcal{C}$ at $c$. Show also that the horocycle is orthogonal to all the geodesics in $\mathcal{D}_P$ that pass through $c$.

11.4.1 Which pairs of hyperbolic lines in $\mathcal{H}$ are parallel? Ultraparallel?

11.4.2 Let $l$ be the imaginary axis in $\mathcal{H}$. Show that, for any $R > 0$, the set of points that are a distance $R$ from $l$ is the union of two half-lines passing through the origin, but that these half-lines are *not* hyperbolic lines. This contrasts with the situation in Euclidean geometry, in which the set of points at a fixed distance from a line is a pair of lines.

11.4.3 Let $a$ and $b$ be two distinct points in $\mathcal{D}_P$, and let $0 < \mathcal{A} < \pi$. Show that the set of points $c \in \mathcal{D}_P$ such that the hyperbolic triangle with vertices $a, b$ and $c$ has area $\mathcal{A}$ is the union of two segments of lines or circles, but that these are not hyperbolic lines. Note that this equal-area property could be used to characterize lines in Euclidean geometry.

11.4.4 A triangle in $\mathcal{D}_P$ is called *asymptotic*, *biasymptotic* or *triasymptotic* if it has one, two or three vertices on the boundary of $\mathcal{D}_P$, respectively (so that one, two or three pairs of sides are parallel). Note that such a triangle always has at least two sides of infinite length.
(i) Show that any triasymptotic triangle has area $\pi$.
(ii) Show that the area of a biasymptotic triangle with angle $\alpha$ is $\pi - \alpha$. Show that such a triangle exists for any $\alpha$ with $0 < \alpha < \pi$.
(iii) Show that the area of an asymptotic triangle with angles $\alpha$ and $\beta$ is $\pi - \alpha - \beta$. Express the length of the finite side of the triangle in terms of $\alpha$ and $\beta$.

11.4.5 Prove that:
(i) If two asymptotic triangles have the same angles (interpreting the angle at the vertex on the boundary as zero), they are congruent.
(ii) The same result as in (i) holds for biasymptotic triangles.
(iii) Any two triasymptotic triangles are congruent.

11.4.6 It is a theorem of Euclidean geometry that the altitudes of a triangle meet at a single point (the altitudes are the straight lines through the vertices perpendicular to the opposite sides). By considering first a suitable biasymptotic triangle, show that the corresponding result in hyperbolic geometry is *not* true.

11.5.1 Prove Eq. (11.8).

11.5.2 Extend the definition of cross-ratio in the obvious way to include the possibility that one of the points is equal to $\infty$, e.g. $(\infty, b; c, d) = (b - d)/(b - c)$. Show that, if $M : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is a Möbius transformation, then

$$(M(a), M(b); M(c), M(d)) = (a, b; c, d) \text{ for all distinct points } a, b, c, d \in \mathbb{C}_\infty.$$

Show, conversely, that if $M : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is a bijection satisfying this condition, then $M$ is a Möbius transformation.

11.5.3 Use the preceding exercise to show that, if $(a, b, c)$ and $(a', b', c')$ are two triples of distinct points of $\mathbb{C}_\infty$, there is a unique Möbius transformation $M$ such that $M(a) = a'$, $M(b) = b'$ and $M(c) = c'$.

11.5.4 Let $a, b \in \mathbb{C}_\infty$ and let $d$ be the spherical distance between the points of $S^2$ that correspond to $a, b$ under the stereographic projection map $\Pi$ (Example 6.3.5). Show that

$$-\tan^2 \frac{1}{2} d = \left( a, -\frac{1}{\bar{a}}; b, -\frac{1}{\bar{b}} \right).$$

11.5.5 Show that, if $\mathcal{R}$ is the reflection in a line passing through the origin, then $\mathcal{K}\mathcal{R} = \mathcal{R}\mathcal{K}$. Deduce that $\mathcal{R}$ is an isometry of $\mathcal{D}_K$.

11.5.6 Show that the isometries of $\mathcal{D}_K$ are precisely the composites of (finitely-many) perspectivities and reflections in lines passing through the origin.

11.5.7 Show that the angle between two curves in $\mathcal{D}_K$ that intersect at the origin is the same as the Euclidean angle of intersection.

11.5.8 Show that points $a, b, c, d \in \mathbb{C}_\infty$ lie on a Circle if and only if $(a, b; c, d)$ is real (see Appendix 2 for the definition of a Circle (capital C!)).

11.5.9 If $\lambda = (a, b; c, d)$, show that the cross-ratio obtained by taking the same points $a, b, c, d$ in a different order has one of the six values $\lambda$, $1/\lambda$, $1 - \lambda$, $1/(1 - \lambda)$, $\lambda/(1 - \lambda)$, $(1 - \lambda)/\lambda$.

# Chapter 12

12.1.1 Show that the Gaussian curvature of a minimal surface is $\leq 0$ everywhere, and that it is zero everywhere if and only if the surface is an open subset of a plane. We shall obtain a much more precise result in Corollary 12.5.6.

12.1.2 Let $\boldsymbol{\sigma} : U \to \mathbb{R}^3$ be a minimal surface patch, and assume that $\mathcal{A}_{\boldsymbol{\sigma}}(U) < \infty$ (see Definition 6.4.1). Let $\lambda \neq 0$ and assume that the principal curvatures $\kappa$ of $\boldsymbol{\sigma}$ satisfy $|\lambda\kappa| < 1$ everywhere, so that the parallel surface $\boldsymbol{\sigma}^\lambda$ of $\boldsymbol{\sigma}$ (Definition 8.5.1) is a regular surface patch. Prove that

$$\mathcal{A}_{\boldsymbol{\sigma}^\lambda}(U) \leq \mathcal{A}_{\boldsymbol{\sigma}}(U)$$

and that equality holds for some $\lambda \neq 0$ if and only if $\boldsymbol{\sigma}(U)$ is an open subset of a plane. (Thus, any minimal surface is area-minimizing among its family of parallel surfaces.)

12.1.3 Show that there is no compact minimal surface.

12.1.4 Show that applying a dilation or an isometry of $\mathbb{R}^3$ to a minimal surface gives another minimal surface. Can there be a *local* isometry between a minimal surface and a non-minimal surface?

12.1.5 Show that every umbilic on a minimal surface is a planar point.

12.1.6 Let $\mathcal{S}$ be a parallel surface of a minimal surface (Definition 8.5.1), and let $\kappa_1$ and $\kappa_2$ be the principal curvatures of $\mathcal{S}$. Show that

$$\frac{1}{\kappa_1} + \frac{1}{\kappa_2} = \text{ constant.}$$

12.1.7 Show that no tube (Exercise 4.2.7) is a minimal surface.

12.1.8 Let $\mathcal{S}$ be a minimal surface, let $\mathbf{p} \in \mathcal{S}$ and let $\mathbf{t}$ be any non-zero tangent vector to $\mathcal{S}$ at $\mathbf{p}$. Show that the Gaussian curvature of $\mathcal{S}$ at $\mathbf{p}$ is

$$K(\mathbf{p}) = -\frac{\langle\langle\langle \mathbf{t}, \mathbf{t} \rangle\rangle\rangle}{\langle \mathbf{t}, \mathbf{t} \rangle}$$

(see Exercise 8.2.31).

12.2.1 Show that every helicoid is a minimal surface.

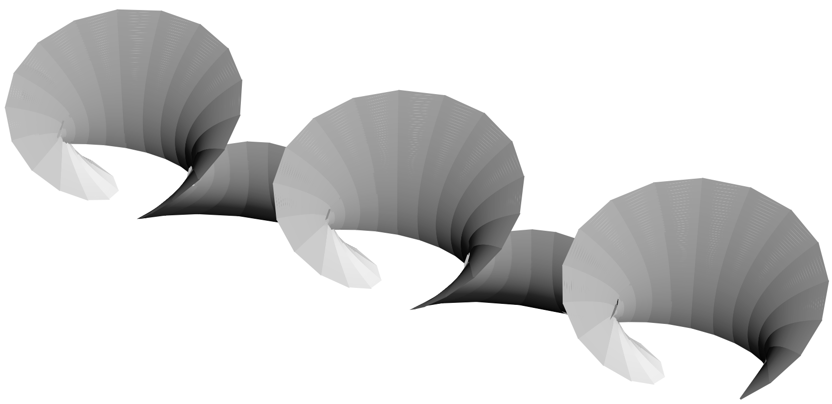
12.2.2 Show that the surfaces $\boldsymbol{\sigma}^t$ in the isometric deformation of a helicoid into a catenoid given in Exercise 6.2.2 are minimal surfaces. (This is 'explained' in Exercise 12.5.4.)

12.2.3 Show that a generalized cylinder is a minimal surface only when the cylinder is an open subset of a plane.

12.2.4 Verify that *Catalan's surface*

$$\boldsymbol{\sigma}(u, v) = \left(u - \sin u \cosh v, 1 - \cos u \cosh v, -4 \sin \frac{u}{2} \sinh \frac{v}{2}\right)$$

is a conformally parametrized minimal surface. (As in the case of Enneper's surface, Catalan's surface has self-intersections, so it is only a surface if we restrict $(u, v)$ to sufficiently small open sets.)



Show that:
(i) The parameter curve on the surface given by $u = 0$ is a straight line.
(ii) The parameter curve $u = \pi$ is a parabola.
(iii) The parameter curve $v = 0$ is a cycloid (see Exercise 1.1.7).
Show also that each of these curves, when suitably parametrized, is a geodesic on Catalan's surface. (There is a sense in which Catalan's surface is 'designed' to have a cycloidal geodesic - see Exercise 12.5.5.)

12.2.5 A *translation surface* is a surface of the form

$$z = f(x) + g(y),$$

where $f$ and $g$ are smooth functions. (It is obtained by "translating the curve $u \mapsto (u, 0, f(u))$ parallel to itself along the curve $v \mapsto (0, v, g(v))$".) Show that this is a minimal surface if and only if

$$\frac{d^2f/dx^2}{1 + (df/dx)^2} = -\frac{d^2g/dy^2}{1 + (dg/dy)^2}.$$

Deduce that any minimal translation surface is an open subset of a plane or can be transformed into an open subset of Scherk's surface in Example 12.2.6 by a translation and a dilation $(x, y, z) \mapsto a(x, y, z)$ for some non-zero constant $a$.

12.2.6 Show that

$$\sin z = \sinh x \sinh y$$

is a minimal surface. It is called *Scherk's fifth minimal surface.*

12.3.1 Let $\mathcal{S}$ be a connected surface whose Gauss map is conformal.
(i) Show that, if $\mathbf{p} \in \mathcal{S}$ and if the mean curvature $H$ of $\mathcal{S}$ at $\mathbf{p}$ is non-zero, there is an open subset of $\mathcal{S}$ containing $\mathbf{p}$ that is part of a sphere.
(ii) Deduce that, if $H$ is non-zero at $\mathbf{p}$, there is an open subset of $\mathcal{S}$ containing $\mathbf{p}$ on which $H$ is constant.
(iii) Deduce that $\mathcal{S}$ is either a minimal surface or an open subset of a sphere.

12.3.2 Show that:
(i) The Gauss map of a catenoid is injective and its image is the whole of $S^2$ except for the north and south poles.
(ii) The image of the Gauss map of a helicoid is the same as that of a catenoid, but that infinitely-many points on the helicoid are sent by the Gauss map to any given point in its image.
(The fact that the Gauss maps of a catenoid and a helicoid have the same image is 'explained' in Exercise 12.5.3(ii).)

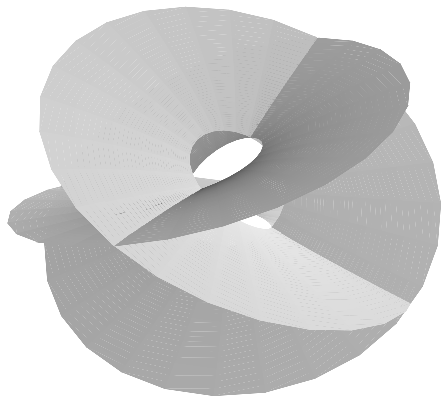
12.4.1 Use Proposition 12.3.2 to give another proof of Theorem 12.4.1 for surfaces $\mathcal{S}$ with nowhere vanishing Gaussian curvature.

12.4.2 It was shown in Exercise 8.2.9 that

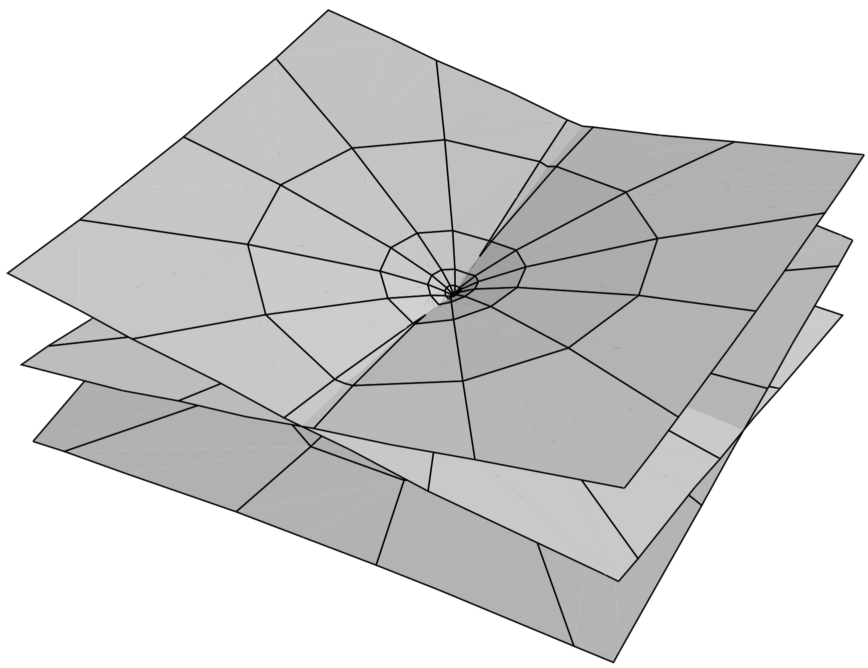
$$y \cos \frac{z}{a} = x \sin \frac{z}{a},$$

where $a$ is a non-zero constant, is a minimal surface. Find a conformal parametrization of this surface.

12.5.1 Find the holomorphic function $\boldsymbol{\varphi}$ corresponding to Enneper's minimal surface given in Example 12.2.5. Show that its conjugate minimal surface coincides with a reparametrization of the same surface rotated by $\pi/4$ around the $z$-axis.



Henneberg: close up



Henneberg: large scale

12.5.2 Find a parametrization of *Henneberg's surface*, the minimal surface corresponding to the functions $f(\zeta) = 1 - \zeta^{-4}$, $g(\zeta) = \zeta$ in Weierstrass's representation. Shown above are a 'close up' view and a 'large scale' view of this surface.

12.5.3 Show that, if $\boldsymbol{\varphi}$ satisfies the conditions in Theorem 12.5.2, so does $a\boldsymbol{\varphi}$ for any non-zero constant $a \in \mathbb{C}$; let $\boldsymbol{\sigma}^a$ be the minimal surface patch corresponding to $a\boldsymbol{\varphi}$, and let $\boldsymbol{\sigma}^1 = \boldsymbol{\sigma}$ be that corresponding to $\boldsymbol{\varphi}$. Show that:
(i) If $a \in \mathbb{R}$, then $\boldsymbol{\sigma}^a$ is obtained from $\boldsymbol{\sigma}$ by applying a dilation and a translation.
(ii) If $|a| = 1$, the map $\boldsymbol{\sigma}(u,v) \mapsto \boldsymbol{\sigma}^a(u,v)$ is an isometry, and the tangent planes of $\boldsymbol{\sigma}$ and $\tilde{\boldsymbol{\sigma}}$ at corresponding points are parallel (in particular, the images of the Gauss maps of $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}^a$ are the same).

12.5.4 Show that if the function $\boldsymbol{\varphi}$ in the preceding exercise is that corresponding to the catenoid (see Example 12.5.3), the surface $\boldsymbol{\sigma}^{e^{it}}$ coincides with the surface denoted by $\boldsymbol{\sigma}^t$ in Exercise 6.2.3.

12.5.5 Let $\boldsymbol{\gamma} : (\alpha, \beta) \to \mathbb{R}^3$ be a (regular) curve in the $xy$-plane, say

$$\boldsymbol{\gamma}(u) = (f(u), g(u), 0),$$

and assume that there are holomorphic functions $F$ and $G$ defined on a rectangle

$$\mathcal{U} = \{u + iv \in \mathbb{C} \mid \alpha < u < \beta,\ -\epsilon < v < \epsilon\},$$

for some $\epsilon > 0$, and such that $F(u) = f(u)$ and $G(u) = g(u)$ if $u$ is real and $\alpha < u < \beta$. Note that (with a dash denoting $d/dz$ as usual),

$$F'(z)^2 + G'(z)^2 \neq 0 \qquad \text{if } \mathfrak{Im}(z) = 0,$$

so by shrinking $\epsilon$ if necessary we can assume that $F'(z)^2 + G'(z)^2 \neq 0$ for all $z \in \mathcal{U}$. Show that:
(i) The vector-valued holomorphic function

$$\boldsymbol{\varphi} = (F', G', i(F'^2 + G'^2)^{1/2})$$

satisfies the conditions of Theorem 12.5.2 and therefore defines a minimal surface $\boldsymbol{\sigma}(u,v)$.
(ii) Up to a translation, $\boldsymbol{\sigma}(u,0) = \boldsymbol{\gamma}(u)$ for $\alpha < u < \beta$.
(iii) $\boldsymbol{\gamma}$ is a pre-geodesic on $\boldsymbol{\sigma}$ (see Exercise 9.1.2).
(iv) If we start with the cycloid

$$\boldsymbol{\gamma}(u) = (u - \sin u, 1 - \cos u, 0),$$

the resulting surface $\boldsymbol{\sigma}$ is, up to a translation, Catalan's surface and we have 'explained' why Catalan's surface has a cycloidal geodesic - see Exercise 12.2.4.

12.5.6 If a minimal surface $\mathcal{S}$ corresponds to a pair of functions $f$ and $g$ in Weierstrass's representation, to which pair of functions does the conjugate minimal surface of $\mathcal{S}$ correspond?

12.5.7 Calculate the functions $f$ and $g$ in Weierstrass's representation for the catenoid and the helicoid.

12.5.8 Let $\mathcal{G}$ be the Gauss map of a minimal surface $\mathcal{S}$ and let $\Pi : S^2 \to \mathbb{C}_\infty$ be the stereographic projection map defined in Example 6.3.5. Show that $\Pi \circ \mathcal{G}$ is the function $g$ in Weierstrass's representation of $\mathcal{S}$.

12.5.9 Find a parametrization of *Richmond's surface*, the minimal surface corresponding to the functions $f(\zeta) = 1/\zeta^2$, $g(\zeta) = \zeta^2$ in Weierstrass's representation. Show that its Gaussian curvature tends to zero as the point $(u, v)$ tends to infinity.

12.5.10 Find the Weierstrass representation of the minimal surface

$$y \cos \frac{z}{a} = x \sin \frac{z}{a},$$

where $a$ is a non-zero constant (see Exercises 8.2.9 and 12.4.2). Hence find the Gaussian curvature of this surface.

# Chapter 13

13.1.1 A surface patch $\boldsymbol{\sigma}$ has Gaussian curvature $\leq 0$ everywhere. Prove that there are no simple closed geodesics on $\boldsymbol{\sigma}$. How do you reconcile this with the fact that the parallels of a circular cylinder are geodesics?

13.1.2 Let $\boldsymbol{\gamma}$ be a unit-speed curve in $\mathbb{R}^3$ with nowhere vanishing curvature. Let $\mathbf{n}$ be the principal normal of $\boldsymbol{\gamma}$, viewed as a curve on $S^2$, and let $s$ be the arc-length of $\mathbf{n}$. Show that the geodesic curvature of $\mathbf{n}$ is, up to a sign,

$$\frac{d}{ds}\left(\tan^{-1} \frac{\tau}{\kappa}\right),$$

where $\kappa$ and $\tau$ are the curvature and torsion of $\boldsymbol{\gamma}$. Show also that, if $\mathbf{n}$ is a *simple closed* curve on $S^2$, the interior and exterior of $\mathbf{n}$ are regions of equal area (*Jacobi's Theorem*).

13.1.3 The vertex of the half-cone

$$x^2 + y^2 = z^2 \tan^2 \alpha, \quad z \geq 0,$$

where the constant $\alpha$ is the semi-vertical angle of the cone, is smoothed so that the cone becomes a regular surface. Prove that the total curvature of the surface is increased by $2\pi(1 - \sin \alpha)$.

13.2.1 Consider the surface of revolution

$$\boldsymbol{\sigma}(u,v) = (f(u)\cos v, f(u)\sin v, g(u)),$$

where $\boldsymbol{\gamma}(u) = (f(u), 0, g(u))$ is a unit-speed curve in the $xz$-plane. Let $u_1 < u_2$ be constants, let $\boldsymbol{\gamma}_1$ and $\boldsymbol{\gamma}_2$ be the two parallels $u = u_1$ and $u = u_2$ on $\boldsymbol{\sigma}$, and let $R$ be the region of the $uv$-plane given by

$$u_1 \leq u \leq u_2, \quad 0 < v < 2\pi.$$

Compute

$$\int_0^{\ell(\boldsymbol{\gamma}_1)} \kappa_g ds, \quad \int_0^{\ell(\boldsymbol{\gamma}_2)} \kappa_g ds \text{ and } \iint_R K d\mathcal{A}_{\boldsymbol{\sigma}},$$

and explain your result on the basis of the Gauss–Bonnet theorem.

13.2.2 Suppose that the Gaussian curvature $K$ of a surface $\mathcal{S}$ satisfies $K \leq -1$ everywhere and that $\boldsymbol{\gamma}$ is a curvilinear $n$-gon on $\mathcal{S}$ whose sides are geodesics. Show that $n \geq 3$, and that, if $n = 3$, the area enclosed by $\boldsymbol{\gamma}$ must be less than $\pi$.

13.2.3 Suppose that the parameter curves of a surface $\mathcal{S}$ are geodesics that intersect at a constant angle. By applying the Gauss-Bonnet theorem to a small curvilinear quadrilateral whose sides are parameter curves, show that $\mathcal{S}$ is flat. Note that this gives another solution of Exercise 10.1.9.

13.3.1 Show that, if a $3 \times 3$ matrix $A$ has rows the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$, then

$$\det(A) = \mathbf{a}.(\mathbf{b} \times \mathbf{c}).$$

13.3.2 Let $n$ be a positive integer. Show that there are smooth functions $\varphi_1, \varphi_2, \ldots, \varphi_{n-1}$ such that
(i) $\varphi_k(t) > 0$ for $\frac{k-1}{n} < t < \frac{k+1}{n}$ and $\varphi_k(t) = 0$ otherwise;
(ii) $\varphi_1(t) + \varphi_2(t) + \cdots + \varphi_{n-1}(t) = 1$ for all $0 < t < 1$.

13.4.1 Show that, if a compact surface $\mathcal{S}$ is diffeomorphic to the torus $T_1$, then

$$\iint_{\mathcal{S}} K \, d\mathcal{A} = 0$$

(cf. Exercise 8.1.9). Can such a surface $\mathcal{S}$ have $K = 0$ everywhere?

13.4.2 Suppose that $\mathcal{S}$ is a compact surface whose Gaussian curvature $K$ is $> 0$ everywhere. Show that $\mathcal{S}$ is diffeomorphic to a sphere. Is the converse of this statement true?

13.4.3 Show that, if $\mathcal{S}$ is the ellipsoid

$$\frac{x^2+y^2}{p^2}+\frac{z^2}{q^2}=1,$$

where $p$ and $q$ are positive constants, then

$$\int\!\!\int_{\mathcal{S}} K\,d\mathcal{A}=4\pi.$$

By computing the above integral directly, deduce that

$$\int_{-\pi/2}^{\pi/2}\frac{pq^2\cos\theta}{(p^2\sin^2\theta+q^2\cos^2\theta)^{3/2}}d\theta=2.$$

13.4.4 What is the Euler number of the compact surface in Exercise 5.4.1?

13.5.1 Prove Proposition 13.5.3.

13.5.2 Show that every triangulation of a compact surface of Euler number $\chi$ by curvilinear triangles has at least $N(\chi)$ vertices.

13.5.3 Show that diffeomorphic compact surfaces have the same chromatic number.

13.5.4 A *cubic map* is a map in which exactly three edges meet at each vertex (like the edges of a cube). Suppose that a cubic map on a surface of Euler number $\chi$ has $c_n$ countries with $n$-edges, for each $n \geq 2$. Show that

$$\sum_{n=2}^{\infty}(6-n)c_n=6\chi.$$

13.5.5 Show that:
(i) A soccer ball must have exactly 12 pentagons (a soccer ball is a cubic map with only pentagons and hexagons).
(ii) If the countries of a cubic map on a sphere are all quadrilaterals or hexagons, there are exactly 6 quadrilaterals.

13.6.1 Let $\boldsymbol{\sigma}(\theta,\varphi)$ be the parametrization of the torus in Exercise 4.2.5. Show that the holonomy around a circle $\theta=\theta_0$ is $2\pi(1-\sin\theta_0)$. Why is it obvious that the holonomy around a circle $\varphi=$ constant is $2\pi$? Note that these circles are *not* simple closed curves on the torus.

13.6.2 Calculate the holonomy around the parameter circle $v=1$ on the cone $\boldsymbol{\sigma}(u,v)=(v\cos u, v\sin u, v)$, and conclude that the converse of Proposition 13.6.5 is false.

13.6.3 In the situation of Proposition 13.6.2, what can we say if $\boldsymbol{\gamma}$ is a closed, but not necessarily simple, curve?

13.6.4 Let $\mathbf{v}$ be a parallel vector field along a unit-speed curve $\boldsymbol{\gamma}$ on a surface $\boldsymbol{\sigma}$, and let $\varphi$ be the oriented angle $\widehat{\dot{\boldsymbol{\gamma}}\mathbf{v}}$. Show that

$$\| \dot{\mathbf{v}} \| = |\kappa_n \cos\varphi + \tau_g \sin\varphi|,$$

where $\kappa_n$ is the normal curvature of $\boldsymbol{\gamma}$ and $\tau_g$ is its geodesic torsion (Exercise 7.3.22).

13.7.1 Let $k$ be a non-zero integer and let $\mathbf{V}(x, y) = (\alpha, \beta)$ be the vector field on the plane given by

$$\alpha + i\beta = \begin{cases} (x + iy)^k & \text{if } k > 0, \\ (x - iy)^{-k} & \text{if } k < 0. \end{cases}$$

Show that the origin is a stationary point of $\mathbf{V}$ of multiplicity $k$.

13.7.2 Show that the definition of a smooth tangent vector field is independent of the choice of surface patch. Show also that a tangent vector field $\mathbf{V}$ on $\mathcal{S}$ is smooth if and only if, for any surface patch $\boldsymbol{\sigma}$ of $\mathcal{S}$, the three components of $\mathbf{V}$ at the point $\boldsymbol{\sigma}(u, v)$ are smooth functions of $(u, v)$.

13.7.3 Show that the Definition 13.7.2 of the multiplicity of a stationary point of a tangent vector field $\mathbf{V}$ is independent of the 'reference' vector field $\boldsymbol{\xi}$.

13.8.1 Show directly that the definitions of a critical point (13.8.1), and whether it is non-degenerate (13.8.2), are independent of the choice of surface patch. Show that the classification of non-degenerate critical points into local maxima, local minima and saddle points is also independent of this choice.

13.8.2 For which of the following functions on the plane is the origin a non-degenerate critical point? In the non-degenerate case(s), classify the origin as a local maximum, local minimum or saddle point.
(i) $x^2 - 2xy + 4y^2$.
(ii) $x^2 + 4xy$.
(iii) $x^3 - 3xy^2$.

13.8.3 Let $\mathcal{S}$ be the torus obtained by rotating the circle $(x - 2)^2 + z^2 = 1$ in the $xz$-plane around the $z$-axis, and let $F : \mathcal{S} \to \mathbb{R}$ be the distance from the plane $x = -3$. Show that $F$ has four critical points, all non-degenerate, and classify them as local maxima, saddle points, or local minima. (See Exercise 4.2.5 for a parametrization of $\mathcal{S}$.)

13.8.4 Show that a smooth function on a torus all of whose critical points are non-degenerate must have at least four critical points.

13.8.5 A rod is attached to a fixed point at one end and a second rod is joined to its other end. Both rods may rotate freely and independently in a vertical plane. Explain why there is a bijection from the torus to the set of possible positions of the two rods.

The potential energy of the rods is a linear combination (with positive coefficients) of the heights of the mid-points of the two rods above some fixed horizontal plane. Show that the corresponding function on the torus has exactly four critical points, all of which are non-degenerate. (These points correspond to the static equilibrium positions of the rods.) Determine whether each critical point is a local maximum, local minimum or saddle point, and verify the result of Theorem 13.8.6.

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